Small Noise Asymptotics, Path-dependence and Thresholds in Presence of Tipping Points

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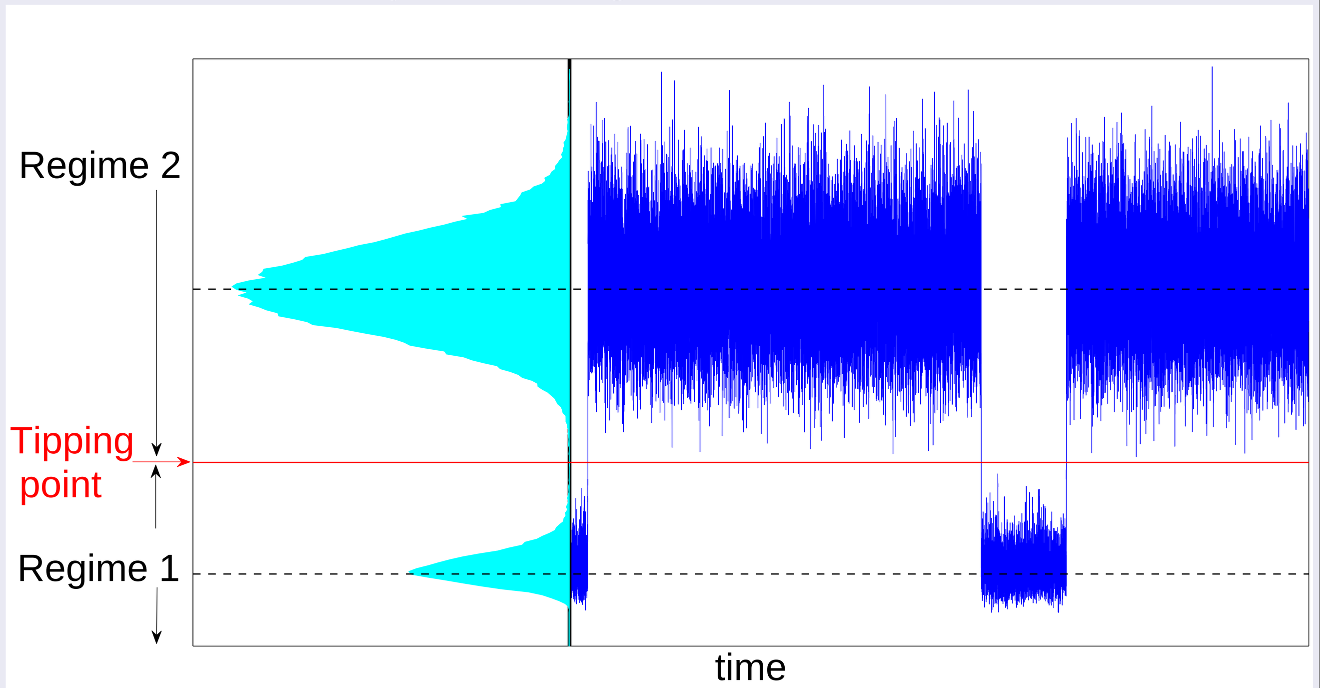
Introduction

Tipping points

Many natural systems exhibit *tipping points*, where a small change in external forces triggers abrupt substantial changes in the state of a system - *regime shift*.



Problem description

Maximize benefit functional

$$\max_u \mathbb{E}_{x_0} \int_0^\infty g(x,u) e^{-\rho t}\, dx$$

$$\text{s.t. } \dot{x} = f(x,u)dt + \sqrt{2\varepsilon\sigma^2(x)}dw,\ x(0) = x_0$$

Value function

$$V(x_0, \varepsilon) = \sup_u \mathbb{E}_{x_0} \int_0^\infty g(x,u) e^{-\rho t}\, dx,$$

Maximized current-value Hamiltonian

$$\mathcal{H}(x,p) = \max_u \{g(x,u) + pf(x,u)\}$$

$V(x,\varepsilon)$ solves the *Hamilton-Jacobi-Bellman equation*

$$\varepsilon\sigma^2(x)V_{xx}(x,\varepsilon) + \mathcal{H}(x, V_x(x,\varepsilon)) - \rho V(x,\varepsilon) = 0.$$

Escape time

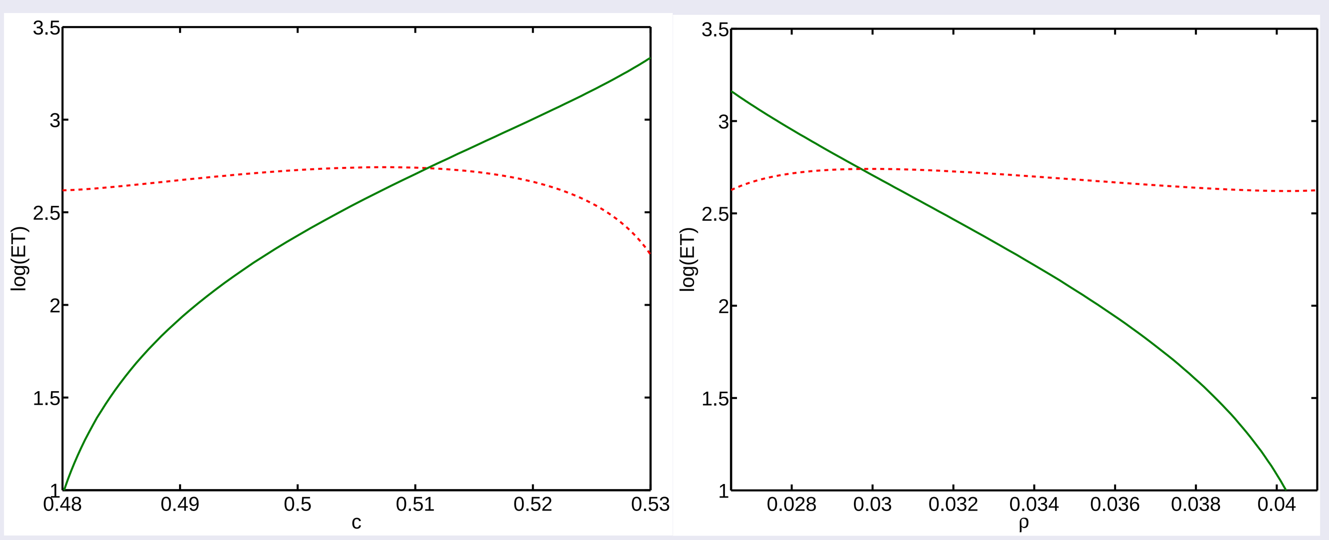


Figure: Escape time from clean(solid) and turbid(dashed) regimes w.r.t. pollution costs c (left) and discount rate ρ (right).

Main result:

We develop an approximation method for stochastic optimal control problems with tipping points. It is based on singular perturbation and matched asymptotic expansion methods. Our method is computationally effective and relatively precise, which allows to use it to perform bifurcation analysis.

The stochastic lake model

$$\max_u \mathbb{E}_{x_0} \int_0^\infty \left(\log u - cx^2\right) e^{-\rho t}\, dt$$

$$\text{s.t. } dx = \left(u - bx + \frac{x^2}{1+x^2}\right) dt + \sqrt{2\varepsilon x^2}dw, x(0) = x_0.$$

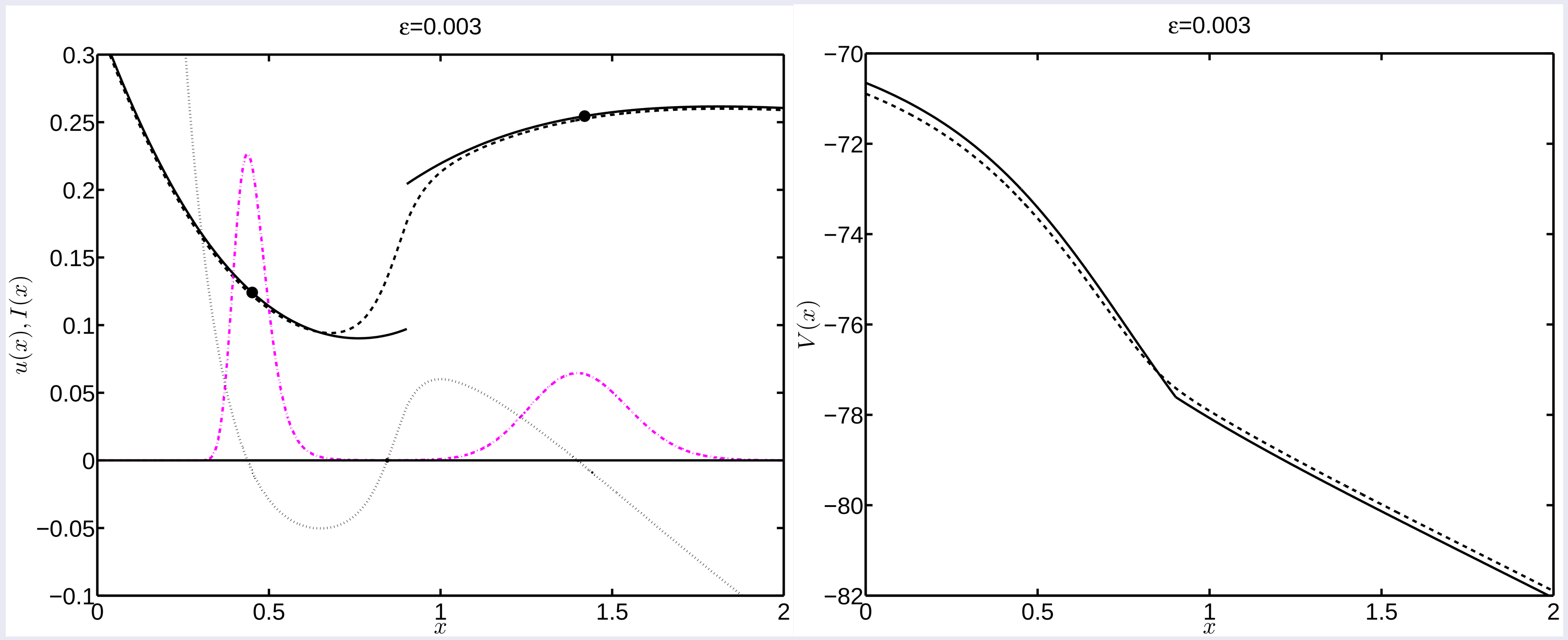


Figure: Optimal control (left) and value functions (right) of the stochastic lake model (dashed) and the corresponding deterministic model (solid). Parameter values: $b = 0.65, c = 0.5, \rho = 0.03, \varepsilon = 0.003$.

Stochastic bifurcations

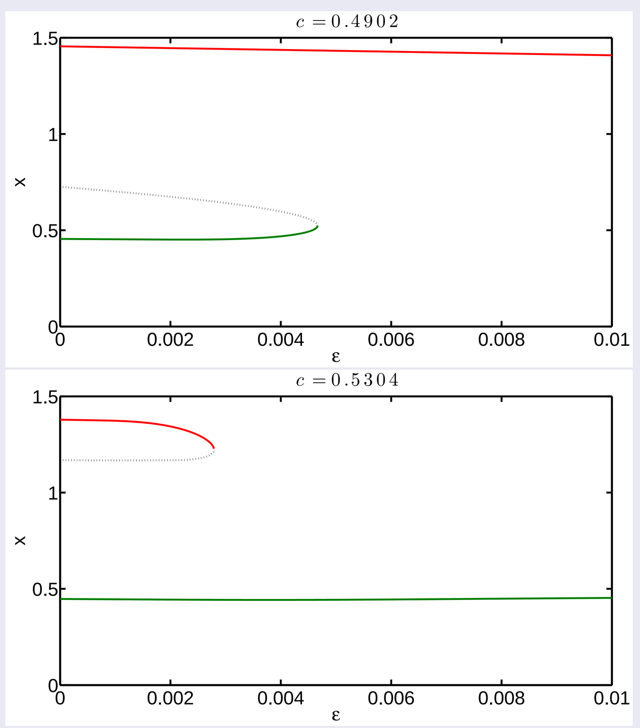


Figure: Bifurcation diagrams for low (top) and high (bottom) pollution costs with respect to the noise intensity ε.

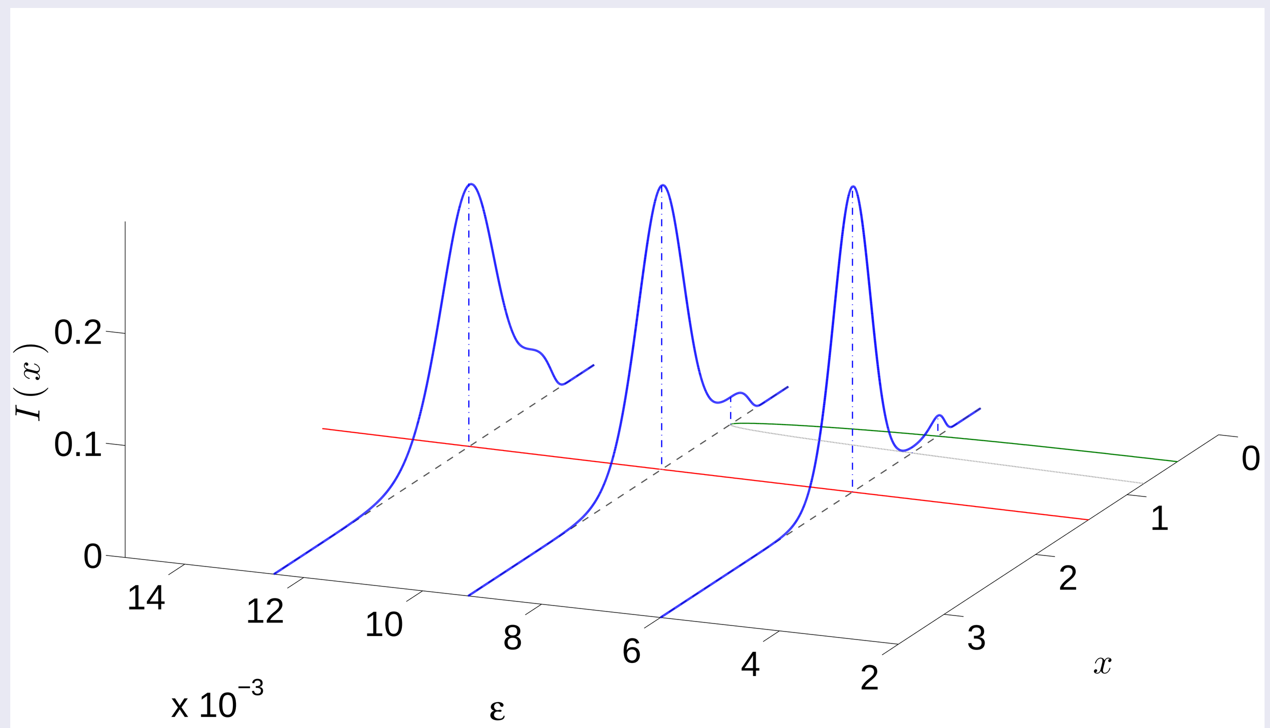


Figure: Bifurcations of transformation invariant function with respect to the noise intensity ε. Parameter values: $b = 0.65, c = 0.5, \rho = 0.03$.

Transformation invariant function

Processes with constant σ.

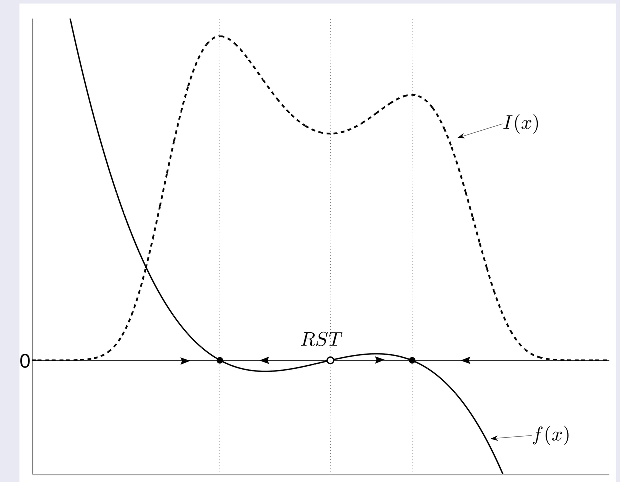
$\phi(x)$ - is stationary PDF of the process

$$dx = f(x)dt + \sigma(x)dw$$

The function

$$I(x) = \sigma(x)\phi(x)$$

is invariant under coordinate transformations.



Stochastic bifurcation of the process x - a change in the number of critical points of $I(x)$.

Approximation method

Outer expansion

If $V^{\text{out}}(x,\varepsilon) = V^{\text{out},k}(x,\varepsilon) + o(\varepsilon^k) = \sum_{j=0}^k \varepsilon^j v_j(x) + o(\varepsilon^k)$ then $v_j(x)$ satisfy equations

$$\mathcal{H}(x, v_0'(x)) = \rho v_0(x),$$
$$\mathcal{H}_p(x, v_0'(x))v_1'(x) = \rho v_1(x) - \sigma^2(x)v_0''(x),$$
$$\mathcal{H}_p(x, v_0'(x))v_2'(x) = \rho v_2(x) - \sigma^2(x)v_1''(x) - \frac{1}{2}\mathcal{H}_{pp}(x, v_0'(x))(v_1'(x))^2$$
$$\ldots$$

Inner expansion

Introduce the local variable

$$\xi = \frac{x - \hat{x}}{\varepsilon}$$

where $\hat{x}-$ *indifference point*, then

$$V^{\text{in}}(\hat{x} + \varepsilon\xi, \varepsilon) = V^{\text{in},k}(\hat{x} + \varepsilon\xi, \varepsilon) + o(\varepsilon^k) = \sum_{j=0}^k \varepsilon^k W_k(\xi) + o(\varepsilon^k)$$
$$= \hat{v} + \varepsilon W_1(\xi) + \ldots + \varepsilon^k W_k(\xi) + o(\varepsilon^k)$$

Functions $W_j(\xi)$ satisfy

$$\sigma^2(\hat{x})W_1'' + \mathcal{H}(\hat{x}, W_1') - \rho\hat{v} = 0$$
$$\sigma^2(\hat{x})W_2'' + \mathcal{H}_p(\hat{x}, W_1')W_2' = -(\sigma^2)'(\hat{x})W_1''\xi - \mathcal{H}_x(\hat{x}, W_1')\xi - \rho W_1$$
$$\ldots$$

Matching and Composite expansion

Outer and inner expansions are matched by equating them in the transition layer. A composite expansion is formed by adding the expansions

$$V^{i,k}(x,\varepsilon) = V^{\text{out},i,k}(x) + V^{\text{inner},k}\left(\frac{x - \hat{x}}{\varepsilon}\right) - \Gamma^{i,k}\left(\frac{x - \hat{x}}{\varepsilon}\right),$$

and subtracting the common part

$$\Gamma^{i,k}\left(\frac{x - \hat{x}}{\varepsilon}\right) = \sum_{j=0}^k \varepsilon^j \sum_{\ell=0}^{k-j} \frac{1}{\ell!} \left(v_j^i\right)^{(\ell)}(\hat{x})\,(x - \hat{x})^\ell$$

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