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What is the Shape of Noise Near a Tipping Point?

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Problem summary

Nonlinear dynamical systems, which include models of the Earth's climate, financial markets and complex ecosystems, often undergo abrupt transitions that lead to radically different behavior. The ability to predict such qualitative and potentially disruptive changes is an important problem with far-reaching implications. In this work we investigate the detection of bifurcation-induced tipping points through a novel, persistent homology-based framework.

Topological detection of tipping

- Let $\frac{dx}{dt} = f_\lambda(x)$ be a continuous, codimension-1 dynamical system.
- $\lambda = \lambda(t)$ is a slowly changing parameter such that the system undergoes a bifurcation at $\lambda = \lambda^*$.
- **Goal:** Develop a robust topological technique to detect critical transitions in dimensions ≥ 1.

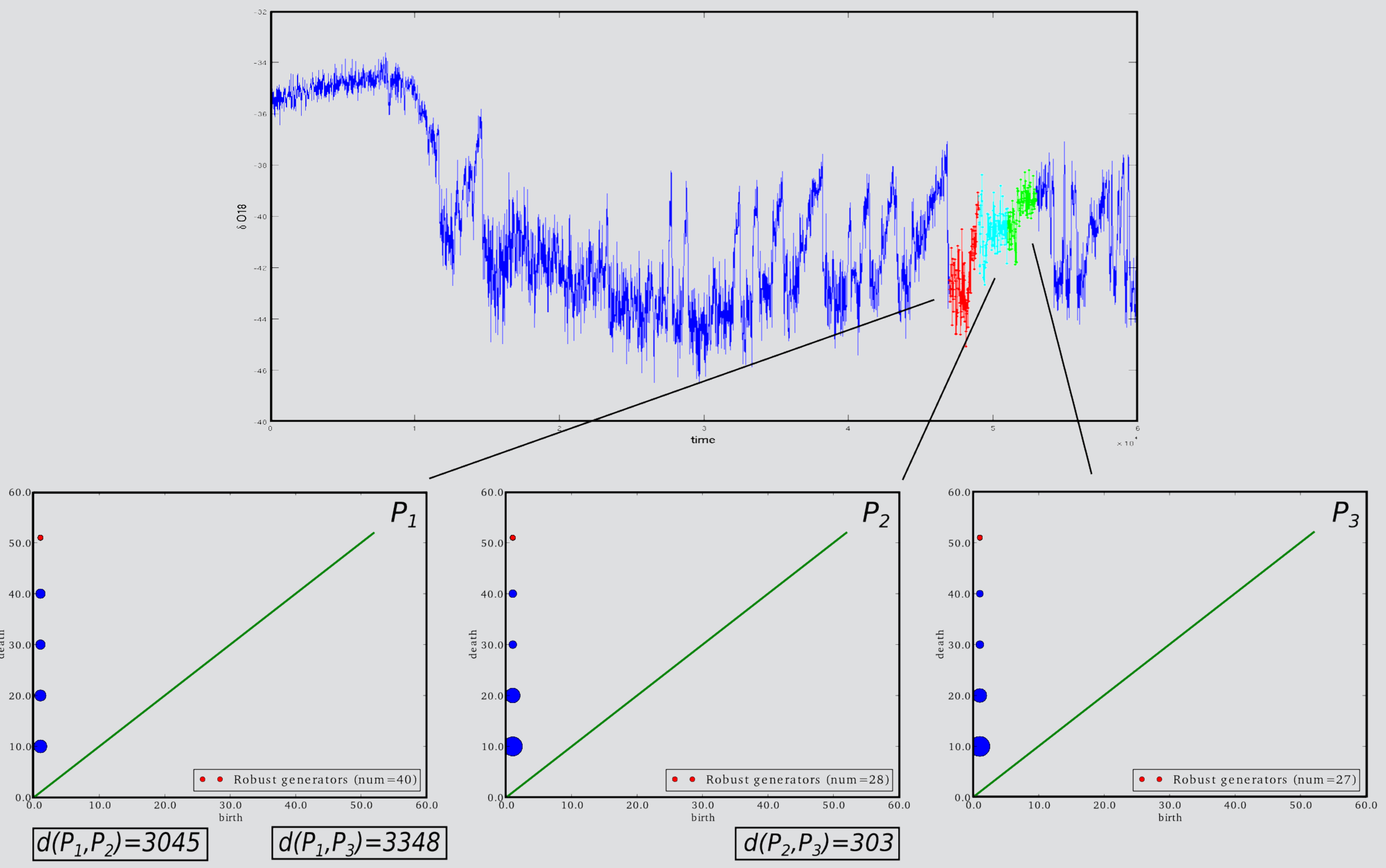
Methodology

- $dx = f_\lambda(x(t))dt + \sigma dW_t$, with sufficiently small σ.
- Suitably large sliding window $W(t)$ along trajectories in $\mathbb{R}^n$.
- Assign to each window the persistence diagram: $W(t) \mapsto P(t)$.

Then $d(P(t + \Delta t), P(t)) > C(\sigma)$ for $\lambda(t) \approx \lambda^*$.

Greenland ice core

Time series of $\delta^{18}O$ data (^{18}O : ^{16}O) have been used to understand past transitions in the Earth's climate. The transition is detectable:



The distance of P_1 (red) from diagrams P_2 (cyan) and P_3 (green) indicates an abrupt change in the system corresponding to critical slowing down.

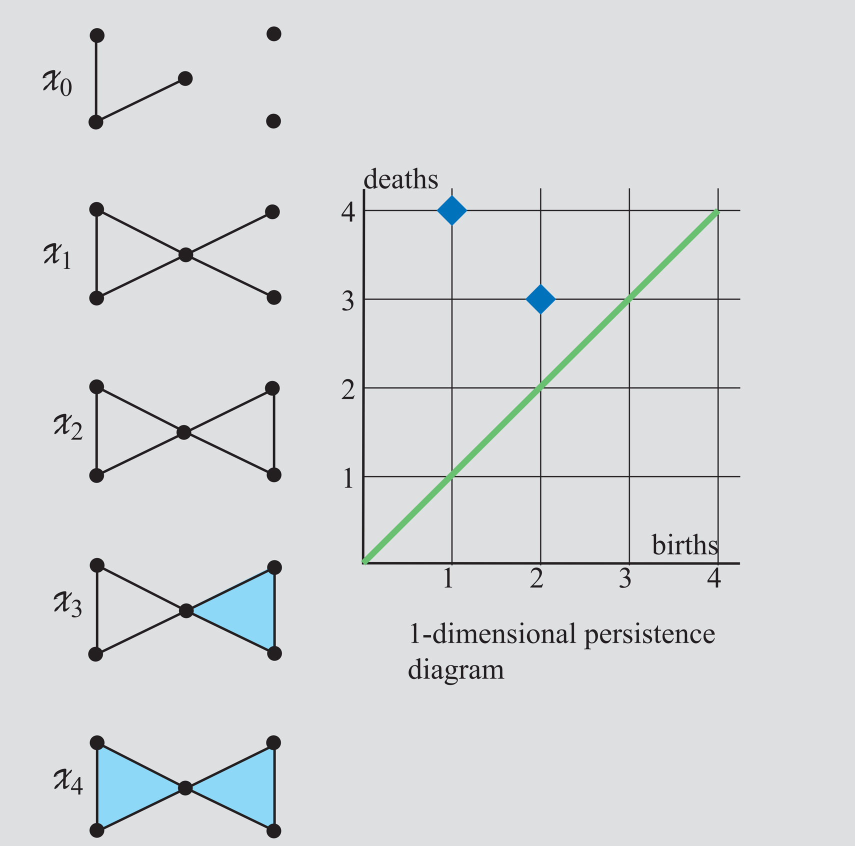
Persistent homology

Topological structures can be inferred from noisy data using *persistent homology*.

- Use the data points to construct filtrations of simplicial complexes:
  - Rips filtration: $\forall a > 0 \rightsquigarrow \mathcal{X}_a$ – simplicial complex.
    - 0-simplices: points $x \in X$
    - k-simplices: $(k+1)$-tuples of $\{x_j\}_{j=0,\ldots,k}$ with $d(x_j, x_{j'}) < a$.
- For $\alpha \in H_k(\mathcal{X}_a)$ — $\exists b_\alpha \leq a \leq d_\alpha$ *birth* time and *death* time of α (w/ multiplicity).
- Persistence diagram $P(X)$ of $\{\mathcal{X}_a\}_{a\geq 0}$ — set of points $(b_\alpha, d_\alpha) \in \mathbb{R}^2$ plus diagonal set.
- "Noise" in the data $\rightsquigarrow$ points close to the diagonal set.
- Persistence diagrams are robust (stable to measurement errors).
  - Persistence diagrams form a metric space with the Wasserstein metric.
  - $X \mapsto P(X)$ is a Lipschitz function.
- If M is a manifold and $S \subset M$, then $d(P(M), P(S)) \leq H(M, S)$ [Hausdorff distance].
- If $S = X$, a noisy approximation of M (a SDE), then $H(M, S) \propto \sigma$, the amplitude of the noise.

If there is a B-tipping point corresponding to a change in the topology of an attractor, then *abrupt* changes between persistence diagrams will detect it.

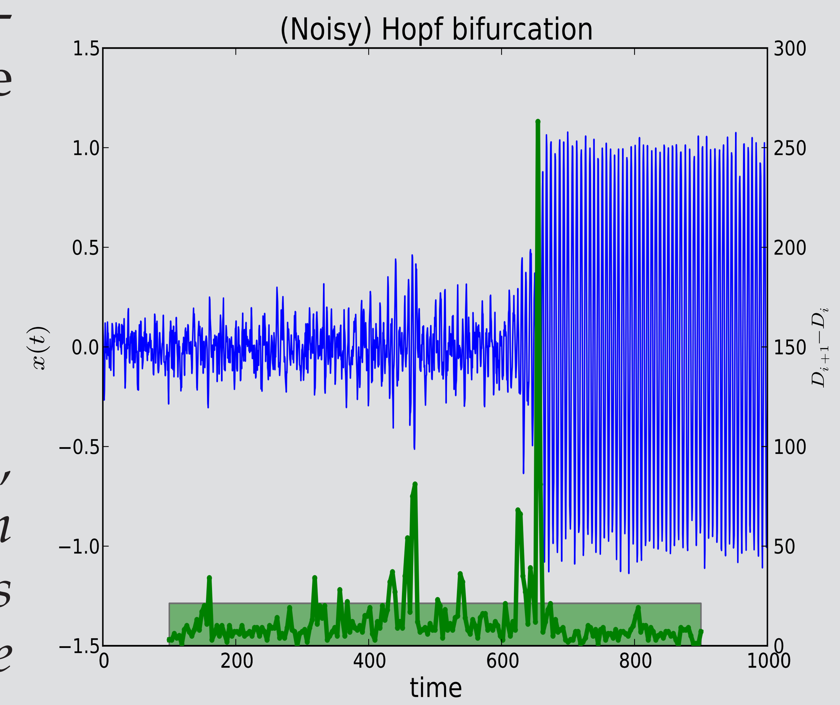
Example of a point cloud, X, and corresponding persistence diagram, $P(X)$.



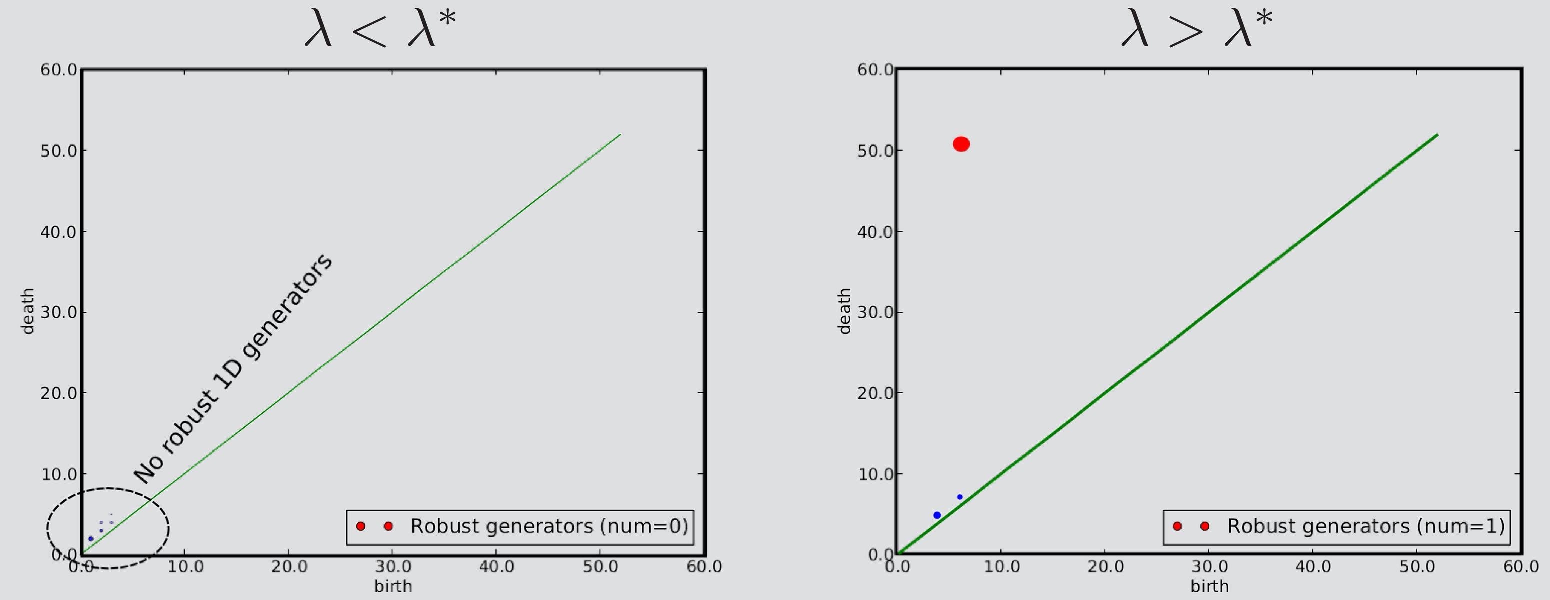
Subcritical Hopf bifurcation

We analyze the (stochastic) subcritical Hopf bifurcation in the plane:

$$dx = f(x, y, \lambda(t))dt + \sigma dW_t$$
$$dy = g(x, y, \lambda(t))dt + \sigma dW_t.$$



Subcritical Hopf bifurcation (blue), y-coordinate. The distance between persistence diagrams (green) detects the change in the topology of the manifold at $t \approx 650 \rightsquigarrow \lambda^*$.

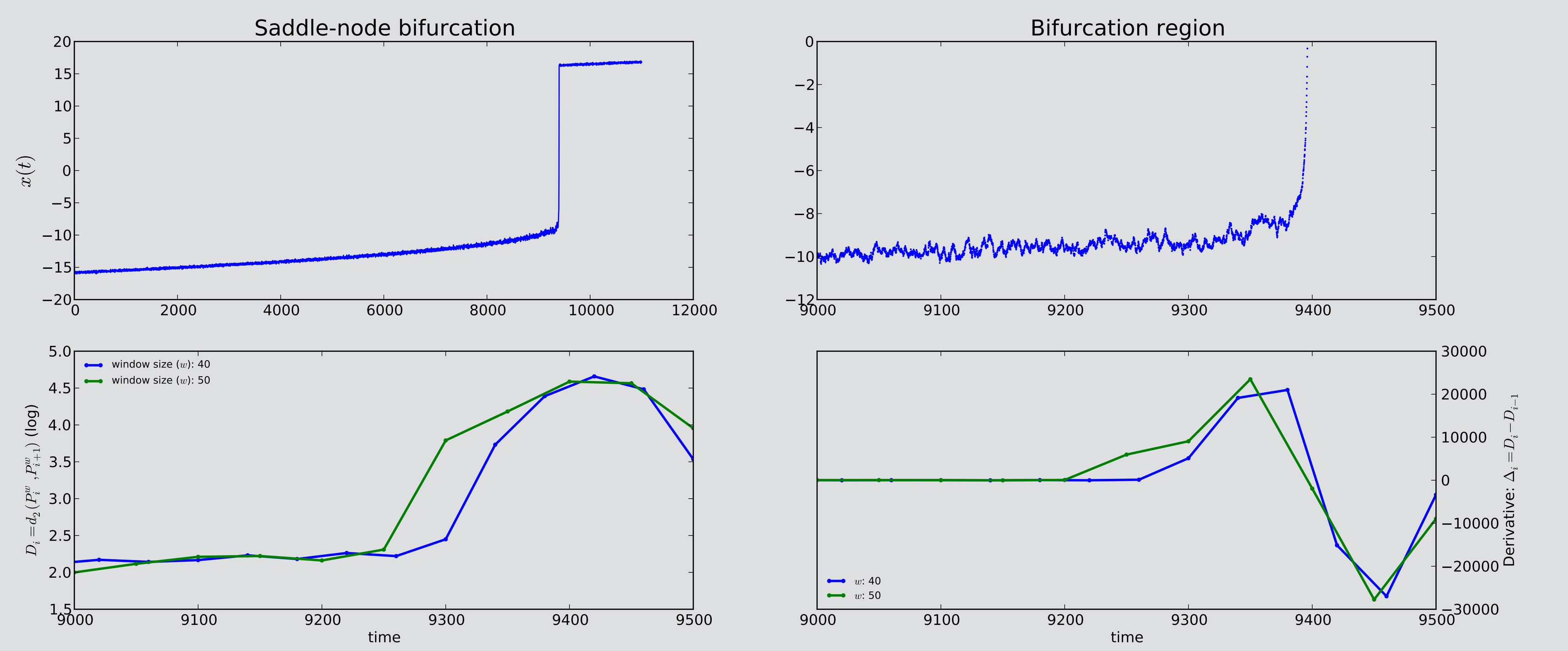


Persistence diagrams quantify the change in topology across λ^*.

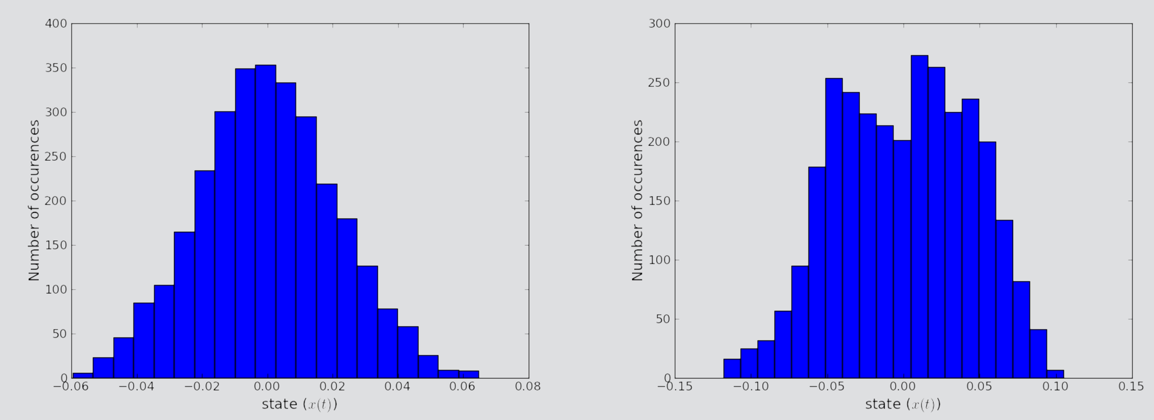
Saddle-node bifurcation

Consider the generic saddle-node bifurcation with noise:

$$dx = -\partial_x U_\mu(x) + \sigma\eta.$$



Top: Time series (left) and bifurcation detail (right). Bottom: Moving average of diagram distances (left); first difference, showing abrupt change prior to bifurcation. Persistence diagrams detect a change in the distrubution of $W(t)$ as $\lambda \rightsquigarrow \lambda^*$.



Distribution (above) of the state variable differs as one approaches the bifurcation indicating. The persistent homology analysis captures this, the same phenomenon found in critical slowing down.

Acknowledgements

- Zomorodian, A. and Carlsson, G. Computing Persistent Homology. Discrete and Computational Geometry (2005)
- Dakos, V., *et al*, Slowing down as an early warning signal for abrupt climate change. PNAS (2008)

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