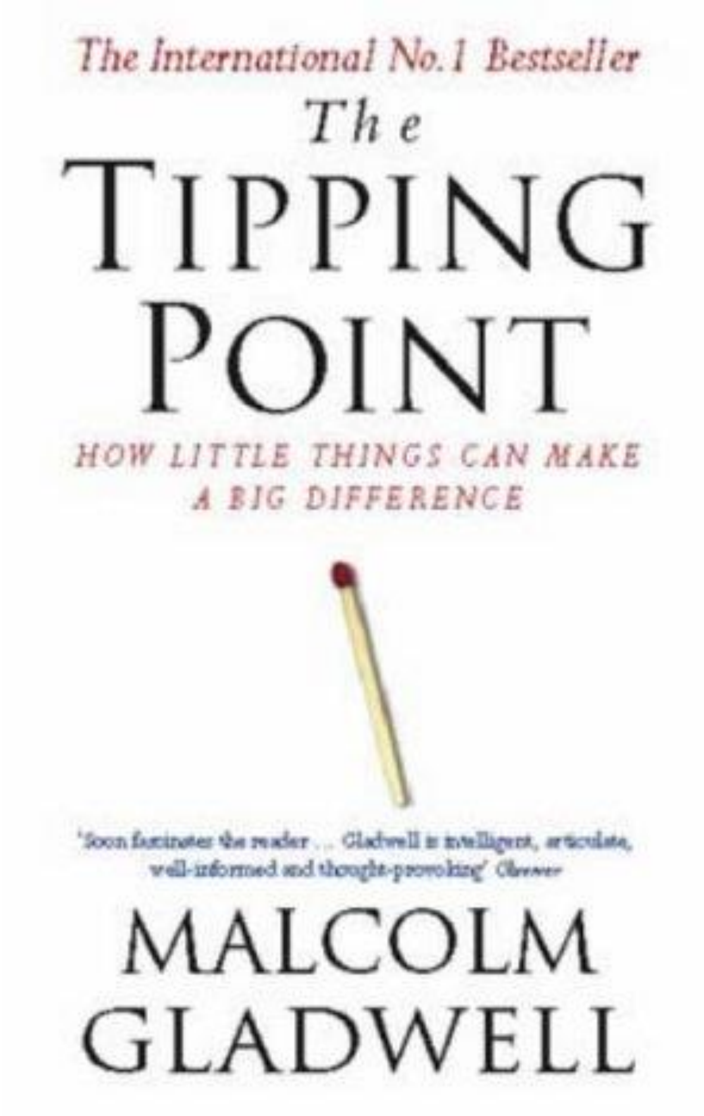
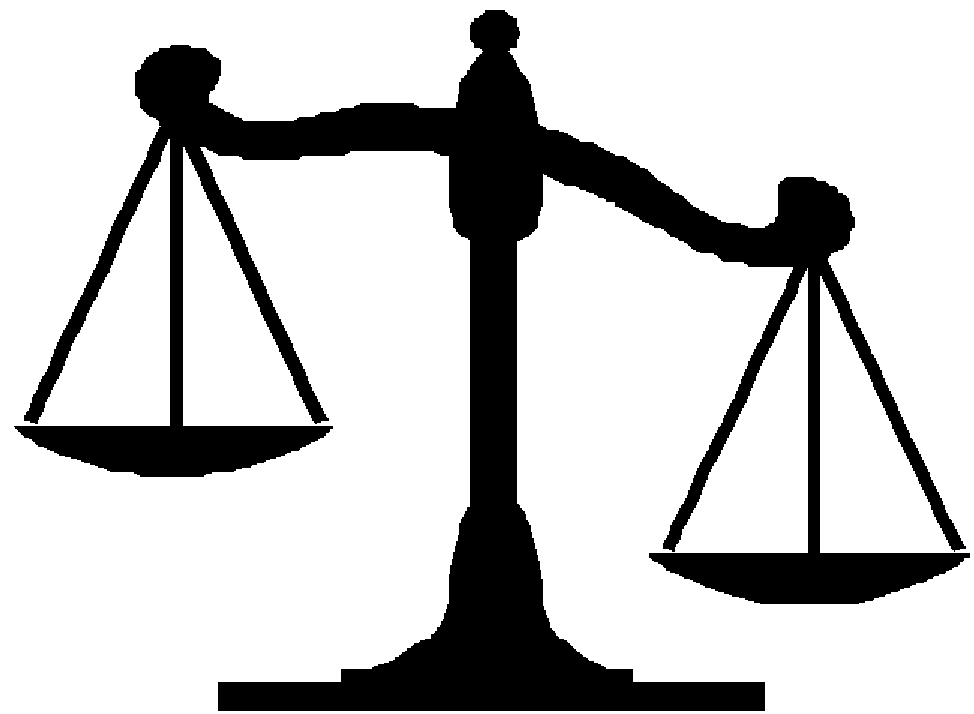
Tipping points and nudges in complex systems

R.S.MacKay,
Director of Mathematical Interdisciplinary Research and of Centre for Complexity Science

THE UNIVERSITY OF WARWICK

Tipping Points





Tipping points in deterministic systems

- Discontinuity of attractor with respect to changes in the system (maybe want to add hysteresis?)
- Use Hausdorff metric on set of compact subsets: smallest ε such that each is in an ε-neighbourhood of the other.
- And measure changes in the system by e.g. C^1 norm of the vector field
- e.g. saddle-node bifurcation produces a jump from node to some other attractor
- But many other types of bifurcation can produce a discontinuity in attractor, e.g. crises

Phases of stochastic systems

- A more appropriate context for many purposes is spatially extended stochastic systems.
- Instead of subsets of state space, consider probability distributions on state space (or even on state space x time).
- The analogue of attractors are “phases”: probability distributions that arise from realisations of systems that have been running since the infinite past.

Reasons

- Chaotic dynamics viewed imprecisely produce a stochastic process.
- Spatial extent is crucial for many complex systems
- Spatially infinite stochastic systems can exhibit more than one phase, without a topological origin.
- Example: NEC majority voter PCA (Vassilyev, Petrovskaya & Piatetski-Shapiro, 1969)

  [Demo by Marina Diakonova]

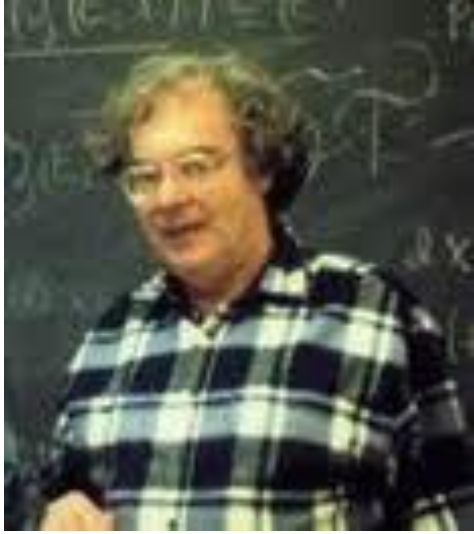
Tipping point for stochastic systems

- Say a tipping point is a discontinuity in a phase with respect to changes in the system (again maybe add hysteresis).
- For this, need a metric on probability distributions; also on systems.

How to measure distance between multivariate probability distributions

- S countable set
- For s in S, (X_s, d_s) Polish (complete separable metric) space of diameter $\leq \Omega$
- $X = \Pi X_s$ with product topology
- ℗ = Borel probabilities on X; want a metric on ℗
- All standard metrics are useless when |S| is large, e.g. "Total variation convergence essentially never occurs for particle systems" (Liggett, 1985). Same for Jeffreys-Jensen-Shannon, Hellinger, Fisher information, projective, transportation (Vasserstein, Kantorovich, Rubinstein) metrics.

Dobrushin metric



- BC = bounded continuous functions f:X➔R
- $\Delta_s(f)$ = sup $(f(x)-f(y))/d_s(x_s,y_s)$
  over x,y in X with $x_r = y_r$ for all $r \neq s$, $x_s \neq y_s$.
- $|f| = \Sigma\, \Delta_s(f)$, *Dobrushin semi-norm*
- F = {f in BC: $|f| < \infty$}, *Dobrushin's functions*
- Z = Borel zero-charge measures μ on X, i.e. μ(X)=0
- |μ| = sup μ(f)/|f| over non-constant f in F
- (Z,|.|) is a Banach space
- For ρ,σ in Ⓟ:
  D(ρ,σ) = |ρ-σ|, *Dobrushin metric,*
  makes Ⓟ a complete metric space (of diameter = sup $\text{diam}_s(X_s)$)

Applications to PCA

- Probability p_s^x on X_s for new state x_s' of site s in S given current state x in X
- Transition probability $p^x = \Pi\ p_s^x$
- Transition operator P on f in BC:

  $(Pf)(x) = p^x(f)$
- Induces P on ρ in Ⓟ by $(\rho P)(f) = \rho(Pf)$
- Want to bound |P| on Z

Dobrushin's dependency matrix

- For ρ,σ probabilities on X_r, define transportation distance $D_r(\rho,\sigma) = \sup (\rho(g)-\sigma(g))/|g|$ over non-constant Lipschitz functions $g: X_r \rightarrow R$, $|g|$ = best Lipschitz constant
- For r,s in S, let $K_{rs} = \sup D_r(p_r^x,p_r^y)/d_s(x_s,y_s)$ over x,y in X with $x_q = y_q$ for all $q \neq s$, $x_s \neq y_s$.
- Then $|P| \leq |K|_\infty = \sup_r \Sigma_s K_{rs}$.
- In particular, $|K|_\infty < 1$ implies P has a unique stationary probability ρ and it attracts exponentially
- e.g. NEC majority voter for λ in $(\frac{1}{3},\frac{2}{3})$
- Same if $|K^t|_\infty \leq Cr^t$ for some r<1: $D(\sigma P^t,\rho) \leq Cr^t D(\sigma,\rho)$
- Exponentially attracting stationary probability is stable to perturbation: $D(\rho,\rho_0) \leq C|\rho_0 (P-P_0)|/(1-r-C|P-P_0|)$, $D(\sigma P^t,\rho) \leq C(r+C|P-P_0|)^t D(\sigma,\rho)$

Strongly dependent PCA

- Toom proved NEC majority voter PCA has at least two phases for small enough error rates.
- Numerically, Bennett and Grinstein (1985) determined a transition curve.
- Find behaviour like catastrophe theory [Demo, Piotr Slowinski]
- Prove jump?
- Revisit Zeeman examples of jumps in social settings?

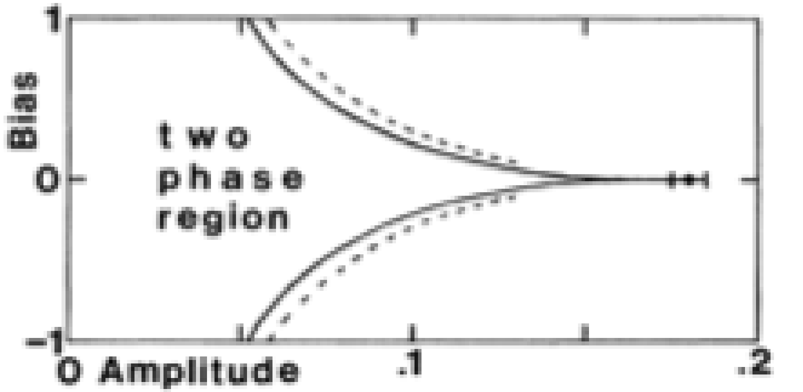
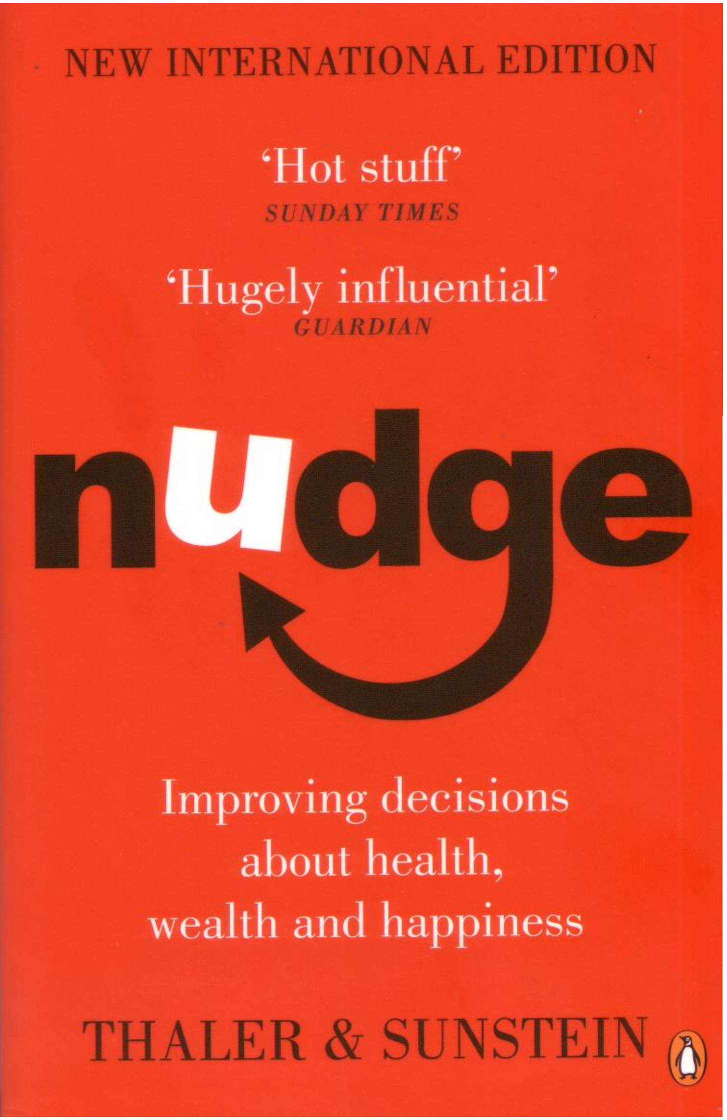


FIG. 1. Phase diagram of the NEC system, for noise parameters p and q, with amplitude $= p + q$ and bias $= (p - q)/(p + q)$.

Nudges



Cabinet Office

Behavioural Insights Team

28 May 2013 — Research and analysis

Applying behavioural insights to charitable giving

Research from the Behavioural Insights Team, or Nudge Unit, shows how charity donations can be increased by using behavioural sciences.

Management of complex systems

- = control of probability distributions (rather than trajectories)
- Two strands:
  - Changing the evolution law
  - Observing the realisation

Response of weakly dependent PCA to time-dependent control

- Suppose time-dependent transition operator P_t, near an exponentially mixing one, then there is a unique time-dependent probability π_t and
  $\pi'_t = \pi_{t-1}P'_{t-1} + \pi_{t-2}P'_{t-2}P_{t-1} + \pi_{t-3}P'_{t-3}P_{t-2}P_{t-1} + \ldots$
- To make sense of the above for PCA, use Dobrushin metric again.

Response of strongly dependent complex systems

- Demonstrations
- Control of phases
  - to encourage transition to a more desirable phase
  - or to prevent transition to an undesirable phase
- Context:
  - infinite system has more than one phase;
  - real system is finite with unique phase but has corresponding metastable phases
- Measure cost of changing transition probability p to p' by $\Sigma_t \sup_x \Sigma_s D_s(p^x{}_s, p'^x{}_s)(t)$
- Want to go from one metastable phase to another with minimum cost.

Attractors and basins

- Phases are mutually singular: for each phase μ there is a subset A_μ of state space with full μ-measure and measure 0 for all other phases
- Define attractor for μ to be smallest such A_μ.
- Define basin B_μ of a phase μ to be the set of states from which realisations go to μ with probability 1.
- Start in attractor A_μ; want to get into B_ν with min cost.

cf Wentzell-Freidlin theory

- Deterministic dynamics plus small noise
- Exists a quasipotential V governing transitions between basins, determined by solution of optimal control problem with respect to the log noise cost.
- Extended to Markov chains on compact manifolds [Kifer, 1990] with some slow transitions.
- Application to noisy globally coupled maps by Hamm, 1999
- Need to extend to spatially extended case.
- Problem of what to count as a neighbourhood of a given state

Conclusion

- Lots of work required, but hope for a theory of tipping points and nudges for spatially extended stochastic systems.