Some New Frontiers in Mathematical Tipping Point Theory

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Overview: Critical Transitions / Tipping Points

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Topics today:

1. Noise-induced transitions for waves in SPDEs.
2. Self-organized criticality in adaptive networks.
3. A different view on large data.

Topic 1: Spatio-Temporal Stochastic Dynamics

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Example: Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).$$

Background - FKPP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).$$

- Model for waves $u = u(x - ct)$ in biology, physics, etc.
- Take $x \in \mathbb{R}$ and localized initial condition $u(x, t = 0)$.
- Many variants e.g. Nagumo PDE.

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Basic propagating front(s):

- $u \equiv 0$ and $u \equiv 1$ are stationary.
- Wave connecting the two states:

$$u(\eta) = u(x - ct), \quad \lim_{\eta \to \infty} u(\eta) = 1, \quad \lim_{\eta \to \infty} u(\eta) = 0.$$

- Propagation into unstable state $u = 0$ since

$$D_u f = D_u[\mu u(1-u)] \quad \Rightarrow D_u f(0) = (\mu - 2u)|_{u=0} > 0.$$

SPDE version of FKPP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) + \sigma g(u)\,\xi(x,t), \qquad \sigma > 0.$$

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Possible choices for 'noise process' $\xi(x,t)$

- white in time $\xi = \dot{B}$, $\mathbb{E}[\dot{B}(t)\dot{B}(s)] = \delta(t-s)$
- space-time white $\xi = \dot{W}$, $\mathbb{E}[\dot{W}(x,t)\dot{W}(y,s)] = \delta(t-s)\delta(x-y)$
- Q-trace-class noise

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Possible choices for 'noise term' $g(u)$

- $g(u) = u$, ad-hoc (Elworthy, Zhao, Gaines,...)
- $g(u) = \sqrt{2u}$, contact-process (Bramson, Durrett, Müller, Tribe,...)
- $g(u) = \sqrt{u(1-u)}$, capacity (Müller, Sowers,...)

Propagation Failure

FKPP SPDE exhibits propagation failure

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) + \sigma g(u)\, \xi(x,t), \qquad g(0) = 0.$$

i.e. solution may get absorbed into $u \equiv 0$.

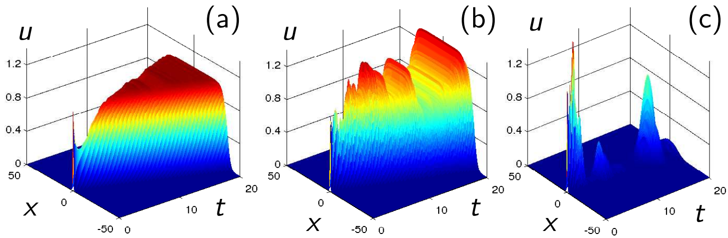

Figure : $g(u) = u$, $\xi = \dot{B}$. (a) $\sigma = 0.02$, (b) $\sigma = 0.3$ and (c) $\sigma = 1.2$.

Scaling near transition: single-point observer statistics:

$$\bar{u} = \frac{1}{T - t_0} \int_{t_0}^{T} u(0, t)\, dt, \quad \Sigma = \left[\frac{1}{T - t_0} \int_{t_0}^{T} (u(0, t) - \bar{u})^2 \; dt \right]^{1/2}.$$

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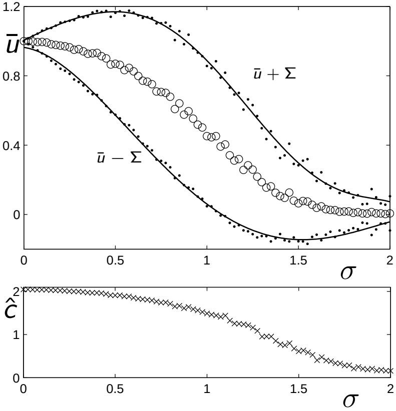

Figure : Average over 200 sample paths; $t \in [10, 20]$.

More on Stochastic FKPP - Further Results

- other noise terms $g(u)$
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- noncompact support for initial condition
- initial transients as warning signs

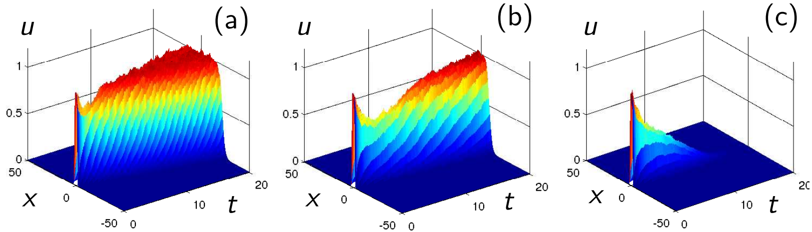

Figure : Nagumo, $g(u) = u$, changing nonlinearity.

Topic 2: Self-organized Criticality in Adaptive Networks

- Adaptive networks with simple local rules can self-organize.
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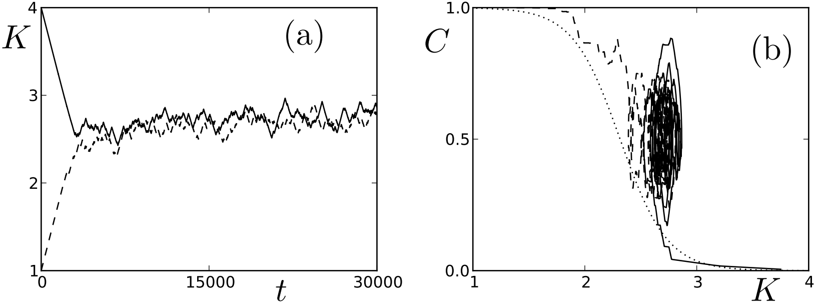

Figure : SOC example. K = average connectivity. C = frozen fraction.

Modified Bornholdt-Rohlf Boolean Network

1. nodes $v_i(t) \in \{\pm 1\}$, directed edges $e_{ij}(t) \in \{-1, 0, +1\}$.
2. dynamical update rule ($t = 0$, random graph), define

$$
\begin{aligned}
f_i(t) &= \sum_j e_{ij}(t) v_j(t) + \mu v_i(t) + \sigma r_i, \;\; \vec{r} \sim \mathcal{N}(0,1) \\
v_i(t+1) &= \begin{cases} \text{sgn}[f_i(t)] & \text{if } f_i(t) \neq 0, \\ v_i(t) & \text{if } f_i(t) = 0. \end{cases}
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3. T_v node dynamics steps, $T_a := \lfloor T_v/2 \rfloor$, measure activity

$$A_i := \frac{1}{T_v - T_a} \left[\sum_{t=T_a}^{T_v - 1} v_i(t) \right].$$

4. topological update rule, choose one node i randomly

$$
\begin{array}{lll}
|A_i| > 1 - \delta & & \text{create an edge } e_{ij}(t) \neq 0, \\
|A_i| \leq 1 - \delta & & \text{delete an edge } e_{ij}(t) = 0.
\end{array}
$$

SOC Ingredients & Observations

Ingredients:

- Large time scale separation $T_v = 1/\epsilon \gg 1$ needed

  topology dynamics $\leftrightarrow$ slow   node dynamics $\leftrightarrow$ fast.

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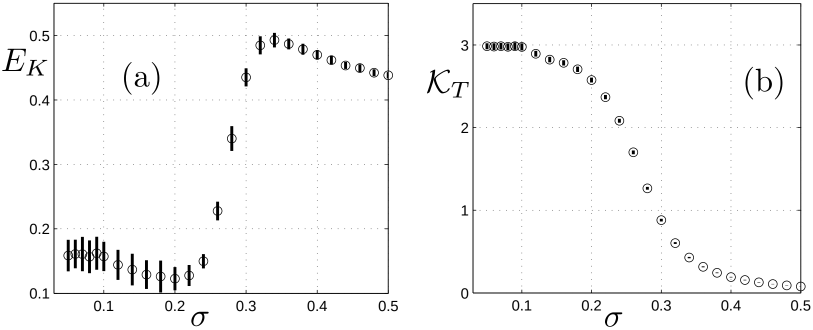
Question: Are there optimal values of (ϵ, σ)?

- Yes for ϵ ('time-scale resonance (TR)')
- Yes for σ ('steady-state stochastic resonance (SSR)')

Back to the Bornholdt-Rohlf Model... and Noise



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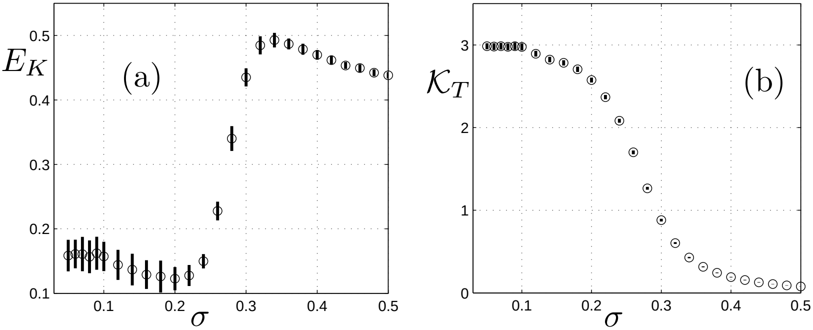

- Non-monotone error, small noise $\rightarrow$ noise optimality.
- SOC tipping, large noise $\rightarrow$ noise-induced phase transition.

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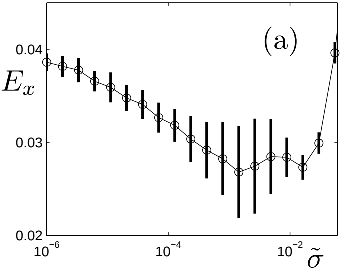

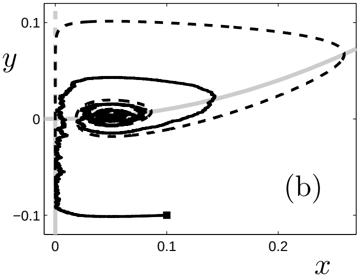

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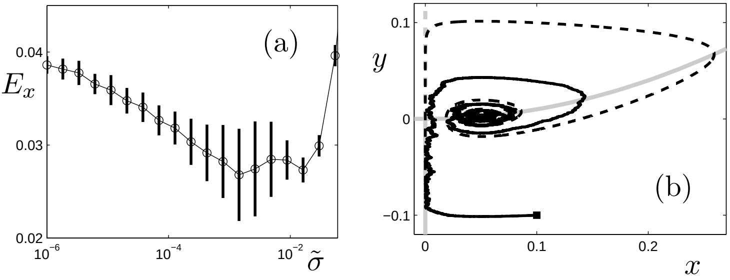

Important new concept - steady-state stochastic resonance (SSR).

Topic 3: Remark on Large Data Sets

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- Approach 2: Abstract scaling laws and genericity.

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3. Social networks, known events (**CK**, Martens, Romero)

References

(1) **CK**, *A mathematical framework for critical transitions: bifurcations, fast-slow systems and stochastic dynamics*, Physica D: Nonlinear Phenomena, Vol. 240, No. 12, pp. 1020-1035, 2011

(2) **CK**, *A mathematical framework for critical transitions: normal forms, variance and applications*, Journal of Nonlinear Science, Vol. 23, No. 3, pp. 457-510, 2013

(3) **CK**, *Warning signs for wave speed transitions of noisy Fisher-KPP invasion fronts*, Theoretical Ecology, Vol. 6, No. 3, pp. 295-308, 2013

(4) **CK**, *Time-scale and noise optimality in self-organized critical adaptive networks*, Physical Review E, Vol. 85, No. 2, 026103, 2012

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(6) **CK**, E.A. Martens and D. Romero, *Critical transitions in social network activity*, arXiv:1307.8250, 2013

(7) **CK**, G. Zschaler and T. Gross, *Early warning signs for critical saddle-escape in complex systems*, preprint, 2013

For more references see also:

- http://www.asc.tuwien.ac.at/∼ckuehn/

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