Math 5490

November 12, 2014

Topics in Applied Mathematics: Introduction to the Mathematics of Climate

Mondays and Wednesdays 2:30 – 3:45

http://www.math.umn.edu/~mcgehee/teaching/Math5490-2014-2Fall/

Streaming video is available at
http://www.ima.umn.edu/videos/
Click on the link: "Live Streaming from 305 Lind Hall".

Participation:
https://umconnect.umn.edu/mathclimate

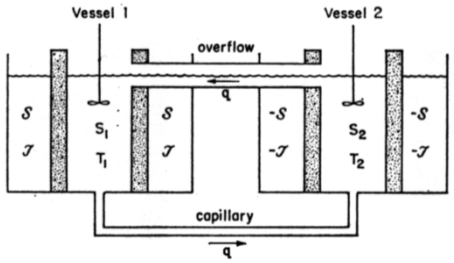
MCRN Mathematics and Climate Research Network

IMA Institute for Mathematics and Its Applications

Dynamical Systems

Stommel's Model

Henry Stommel, *Thermohaline Convection with Two Stable Regimes of Flow*, TELLUS **XII** (1961), 224-230.



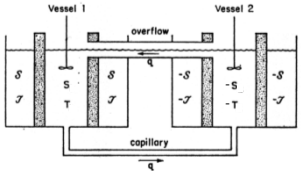
Dynamical Systems

Stommel's Model

$$\frac{dT}{dt} = c(T^* - T) - |2q|T$$
$$\frac{dS}{dt} = d(S^* - S) - |2q|S$$
$$kq = \rho_1 - \rho_2 = \rho_0(-2\alpha T + 2\beta S)$$

$$y = \frac{T}{T^*} \quad x = \frac{S}{S^*} \quad d\tau = cdt$$
$$\delta = \frac{d}{c} \quad R = \frac{\beta S^*}{\alpha T^*} \quad \lambda = \frac{c}{4\rho_0 \alpha T^*} k$$

flow rate $f = \frac{2q}{c}$ — flow resistance



salinity $\frac{dx}{d\tau} = \delta(1-x) - |f|x$

temperature $\frac{dy}{d\tau} = 1 - y - |f|y$

$\lambda f = -y + Rx$

Dynamical Systems

Stommel's Model

$$\frac{dx}{d\tau} = \delta(1-x) - |f|x$$
$$\frac{dy}{d\tau} = 1 - y - |f|y$$
$$\lambda f = -y + Rx$$

Look for equilibria:

$$\delta(1-x_e) - |f|x_e = 0 \qquad x_e = \frac{\delta}{\delta + |f|}$$
$$1 - y_e - |f|y_e = 0 \qquad y_e = \frac{1}{1+|f|}$$

$$\lambda f = -y_e + Rx_e = -\frac{1}{1+|f|} + \frac{R\delta}{\delta+|f|} \equiv \phi(f;R,\delta)$$
$$\boxed{\lambda f = \phi(f;R,\delta)}$$

Solve for f, then solve for equilibrium point.

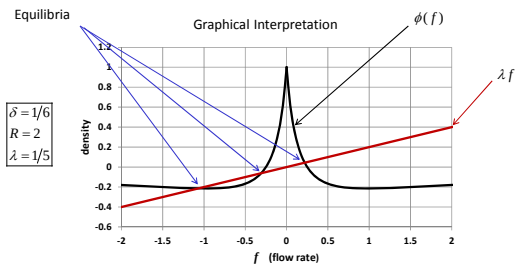
Dynamical Systems

Stommel's Model

$$\lambda f = \phi(f;R,\delta) = \frac{R\delta}{\delta+|f|} - \frac{1}{1+|f|}$$

Equilibria

Graphical Interpretation



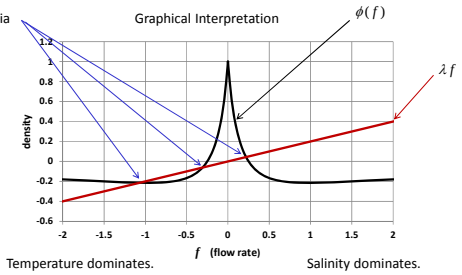
$\delta = 1/6$
$R = 2$
$\lambda = 1/5$

Dynamical Systems

Stommel's Model

Equilibria

Graphical Interpretation



$\delta = 1/6$
$R = 2$
$\lambda = 1/5$

Temperature dominates. capillary flow: cold to warm

Salinity dominates. capillary flow: warm to cold

Dynamical Systems

Stommel's Model

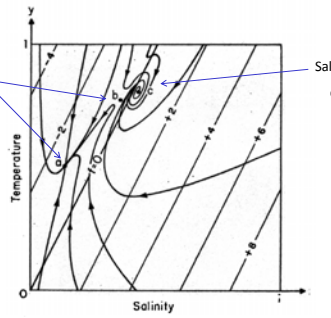
Temperature dominates.

capillary flow: cold to warm

Salinity dominates.

capillary flow: warm to cold

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$



Stommel, TELLUS **XII** (1961)

Dynamical Systems

Stommel's Model

Equilibrium Conditions

$$\delta(1-x_e) - |f| x_e = 0 \qquad x_e = \frac{\delta}{\delta + |f|}$$

$$1 - y_e - |f| y_e = 0 \qquad y_e = \frac{1}{1+|f|}$$

$$\lambda f = -y_e + R x_e = -\frac{1}{1+|f|} + \frac{R\delta}{\delta + |f|}$$

Solve for f,
then solve for equilibrium point.

$$\lambda f (1+|f|)(\delta + |f|) = -(\delta + |f|) + R\delta(1+|f|)$$

$$\lambda f \left(\delta + (1+\delta)|f| + f^2\right) = \delta(R-1) - (1-R\delta)|f|$$

$$\boxed{\lambda f^3 + \lambda(1+\delta) f|f| + \lambda\delta f + (1-R\delta)|f| - \delta(R-1) = 0}$$

Dynamical Systems

Stommel's Model

Equilibrium Conditions: Solving for f

$$\lambda f^3 + \lambda(1+\delta) f|f| + \lambda\delta f + (1-R\delta)|f| - \delta(R-1) = 0$$

Parameters: $\delta = 1/6 \quad R = 2 \quad \lambda = 1/5$

$$\tfrac{1}{5} f^3 + \tfrac{1}{5}(1+\tfrac{1}{6}) f|f| + \tfrac{1}{5}\cdot\tfrac{1}{6} f + (1 - 2\cdot\tfrac{1}{6})|f| - \tfrac{1}{6}(2-1) = 0$$

$$\tfrac{1}{5} f^3 + \tfrac{7}{30} f|f| + \tfrac{1}{30} f + \tfrac{2}{3}|f| - \tfrac{1}{6} = 0$$

Case 1: $f > 0$

$$\tfrac{1}{5} f^3 + \tfrac{7}{30} f^2 + \tfrac{21}{30} f - \tfrac{1}{6} = 0$$

Solve numerically. Only one positive root: $f \approx 0.21909$

Case 2: $f < 0$

$$\tfrac{1}{5} f^3 - \tfrac{7}{30} f^2 - \tfrac{19}{30} f - \tfrac{1}{6} = 0$$

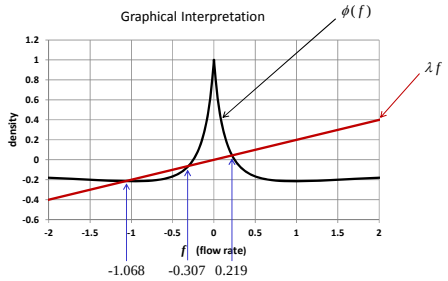
Solve numerically. Two negative roots: $f \approx$ -1.06791, -0.30703

Dynamical Systems

Stommel's Model

Graphical Interpretation

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$



-1.068 -0.307 0.219

Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$

Rest Points

$$x_e = \frac{\delta}{\delta + |f|} = \frac{1}{1+6|f|}, \qquad y_e = \frac{1}{1+|f|}$$

point **a**: $f \approx$ -1.06791:

$$x_e = \frac{1}{1+6|-1.06791|} \approx 0.13500, \qquad y_e = \frac{1}{1+|-1.06791|} \approx 0.48358$$

point **b**: $f \approx$ -0.30703:

$$x_e = \frac{1}{1+6|-1.06791|} \approx 0.35184, \qquad y_e = \frac{1}{1+|-1.06791|} \approx 0.76510$$

point **c**: $f \approx$ 0.21909:

$$x_e = \frac{1}{1+6|-1.06791|} \approx 0.43205, \qquad y_e = \frac{1}{1+|-1.06791|} \approx 0.82028$$

Dynamical Systems

Stommel's Model

Temperature dominates.

capillary flow: cold to warm

$f = 0$

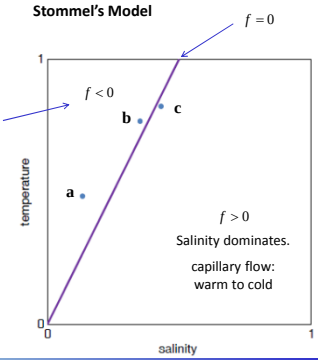
$f < 0$

$f > 0$

Salinity dominates.

capillary flow: warm to cold

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$



Dynamical Systems

Stommel's Model

Structure of Rest Points

$$\dot{x} = \delta(1-x) - |f|x \qquad \lambda f = -y + Rx$$
$$\dot{y} = 1 - y - |f|y$$

Jacobian matrix

$$\begin{bmatrix} -\delta - |f| - x\frac{\partial |f|}{\partial x} & -x\frac{\partial |f|}{\partial y} \\ -y\frac{\partial |f|}{\partial x} & -1 - |f| - y\frac{\partial |f|}{\partial y} \end{bmatrix}$$

$f > 0$: $|f| = -\frac{1}{\lambda}y + \frac{R}{\lambda}x$, $\frac{\partial |f|}{\partial x} = \frac{R}{\lambda}$, $\frac{\partial |f|}{\partial y} = -\frac{1}{\lambda}$

$f < 0$: $|f| = \frac{1}{\lambda}y - \frac{R}{\lambda}x$, $\frac{\partial |f|}{\partial x} = -\frac{R}{\lambda}$, $\frac{\partial |f|}{\partial y} = \frac{1}{\lambda}$

$f > 0$

$$\begin{bmatrix} -\delta - |f| - \frac{R}{\lambda}x & \frac{1}{\lambda}x \\ -\frac{R}{\lambda}y & -1 - |f| + \frac{1}{\lambda}y \end{bmatrix}$$

$f < 0$

$$\begin{bmatrix} -\delta - |f| + \frac{R}{\lambda}x & -\frac{1}{\lambda}x \\ +\frac{R}{\lambda}y & -1 - |f| - \frac{1}{\lambda}y \end{bmatrix}$$

Dynamical Systems

Stommel's Model

Rest Point c

$\delta = 1/6$, $R = 2$, $\lambda = 1/5$

$f \approx 0.21909 > 0, \quad x_e \approx 0.43205, \quad y_e \approx 0.82028$

Jacobian matrix

$$\begin{bmatrix} -\delta - |f| - \frac{R}{\lambda}x & \frac{1}{\lambda}x \\ -\frac{R}{\lambda}y & -1 - |f| + \frac{1}{\lambda}y \end{bmatrix} \approx \begin{bmatrix} -\frac{1}{6} - |0.21909| - \frac{2}{1/5}0.43205 & \frac{1}{1/5}0.43205 \\ -\frac{2}{1/5}0.82028 & -1 - |0.21909| + \frac{1}{1/5}0.82028 \end{bmatrix}$$
$$\approx \begin{bmatrix} -4.70627 & 2.16025 \\ -8.20284 & 2.88233 \end{bmatrix}$$

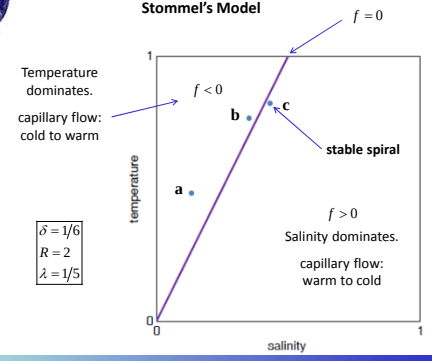
determinant $\approx 4.15521 > 0$

trace $\approx -1.82394 < 0$ ⟵ **stable**

discriminant $\approx -13.29410 < 0$ ⟵ **spiral**

Dynamical Systems

Stommel's Model



Dynamical Systems

Stommel's Model

Rest Point b

$\delta = 1/6$, $R = 2$, $\lambda = 1/5$

$f \approx -0.30703 < 0, \quad x_e \approx 0.35184, \quad y_e \approx 0.76510$

Jacobian matrix

$$\begin{bmatrix} -\delta - |f| + \frac{R}{\lambda}x & -\frac{1}{\lambda}x \\ +\frac{R}{\lambda}y & -1 - |f| - \frac{1}{\lambda}y \end{bmatrix} \approx \begin{bmatrix} -\frac{1}{6} - |-0.30703| + \frac{2}{1/5}0.35184 & -\frac{1}{1/5}0.35184 \\ +\frac{2}{1/5}0.76510 & -1 - |-0.30703| - \frac{1}{1/5}0.76510 \end{bmatrix}$$
$$\approx \begin{bmatrix} 3.04476 & -1.75922 \\ 7.65095 & -5.13250 \end{bmatrix}$$

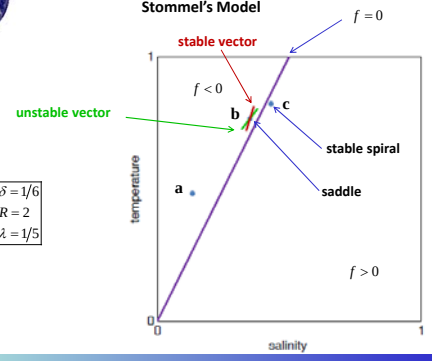
eigenvalue ≈ 7.60883, eigenvector $\approx \begin{bmatrix} 0.61023 \\ 0.79222 \end{bmatrix}$

eigenvalue ≈ -0.28486, eigenvector $\approx \begin{bmatrix} 0.28604 \\ 0.95822 \end{bmatrix}$

saddle

Dynamical Systems

Stommel's Model



Dynamical Systems

Stommel's Model

Rest Point a

$\delta = 1/6$, $R = 2$, $\lambda = 1/5$

$f \approx -1.06791 < 0, \quad x_e \approx 0.13500, \quad y_e \approx 0.48358$

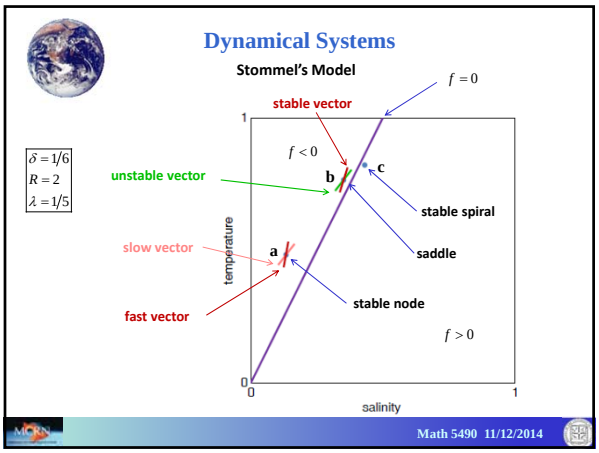
Jacobian matrix

$$\begin{bmatrix} -\delta - |f| + \frac{R}{\lambda}x & -\frac{1}{\lambda}x \\ +\frac{R}{\lambda}y & -1 - |f| - \frac{1}{\lambda}y \end{bmatrix} \approx \begin{bmatrix} -\frac{1}{6} - |-1.06791| + \frac{2}{1/5}0.13500 & -\frac{1}{1/5}0.13500 \\ +\frac{2}{1/5}0.48358 & -1 - |-1.06791| - \frac{1}{1/5}0.48358 \end{bmatrix}$$
$$\approx \begin{bmatrix} 0.11541 & -0.67500 \\ 0.48358 & -4.48581 \end{bmatrix}$$

eigenvalue ≈ -0.76088, eigenvector $\approx \begin{bmatrix} 0.61023 \\ 0.79222 \end{bmatrix}$

eigenvalue ≈ -3.60951, eigenvector $\approx \begin{bmatrix} 0.17831 \\ 0.98398 \end{bmatrix}$

stable node


Dynamical Systems
Stommel's Model
$\delta = 1/6$
$R = 2$
$\lambda = 1/5$
$f = 0$
stable vector
$f < 0$
unstable vector
b
c
stable spiral
slow vector
a
saddle
fast vector
stable node
$f > 0$
temperature
salinity
0
1
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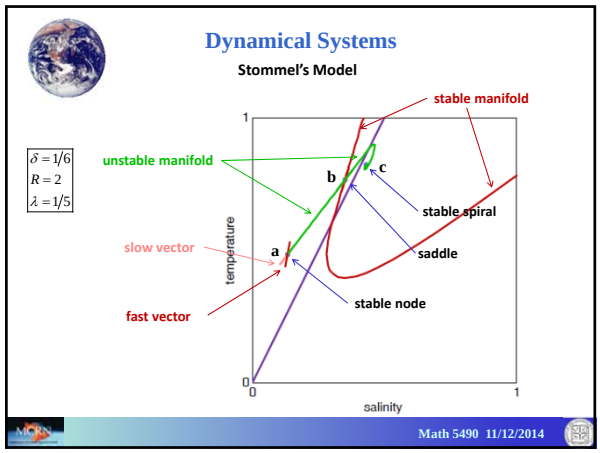
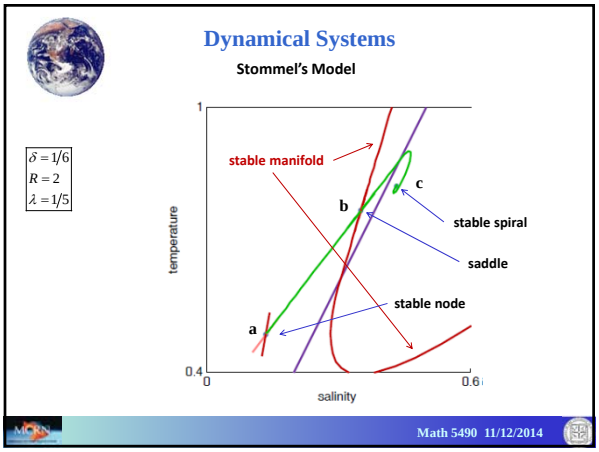
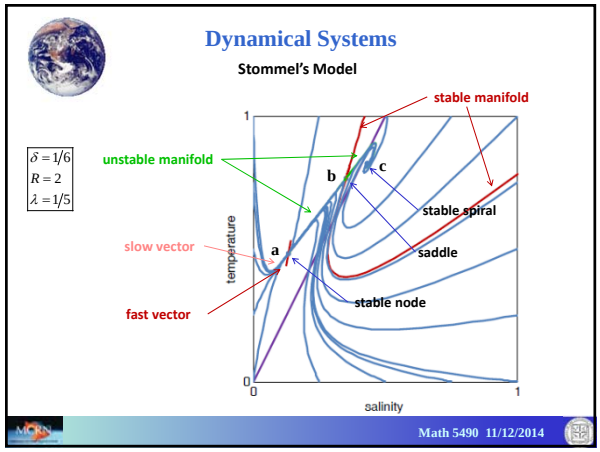

Dynamical Systems
Stommel's Model
$\delta = 1/6$
$R = 2$
$\lambda = 1/5$
stable manifold
unstable manifold
b
c
stable spiral
slow vector
a
saddle
fast vector
stable node
temperature
salinity
0
1
Math 5490 11/12/2014


Dynamical Systems
Stommel's Model
$\delta = 1/6$
$R = 2$
$\lambda = 1/5$
stable manifold
b
c
stable spiral
saddle
stable node
a
temperature
salinity
0.4
0
0.6
1
Math 5490 11/12/2014


Dynamical Systems
Stommel's Model
$\delta = 1/6$
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stable manifold
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c
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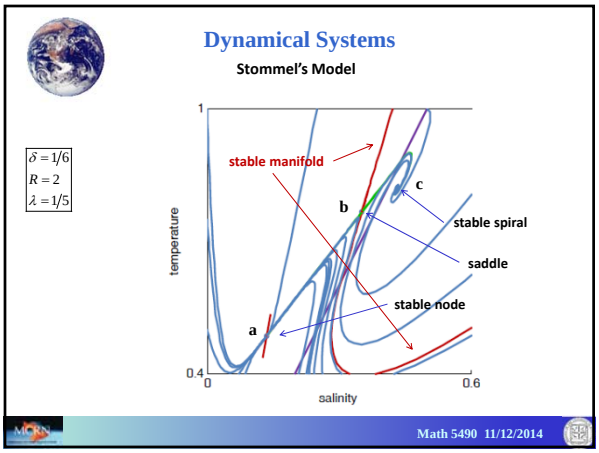
stable node
temperature
salinity
0
1
Math 5490 11/12/2014


Dynamical Systems
Stommel's Model
$\delta = 1/6$
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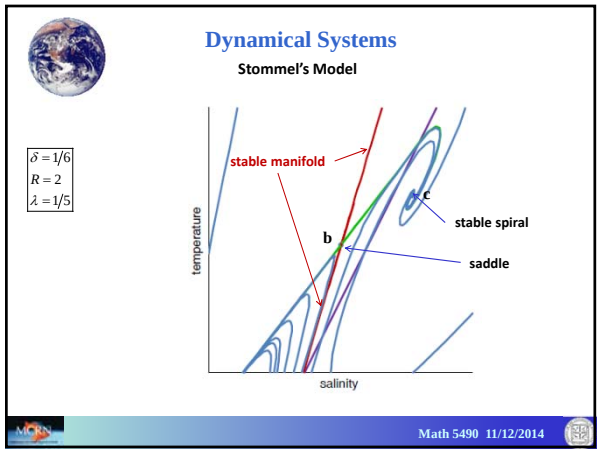
stable manifold
b
c
stable spiral
saddle
stable node
a
temperature
salinity
0.4
0
0.6
1
Math 5490 11/12/2014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Stommel's Model
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stable manifold
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Math 5490 11/12/2014

Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$

stable manifold
unstable manifold
stable spiral
saddle
slow vector
stable node
fast vector

a
b
c

temperature
salinity
0
1

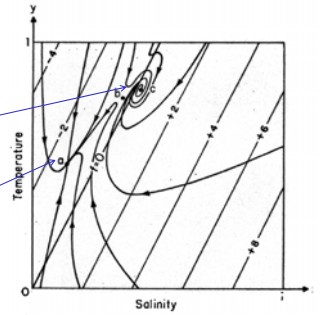
Dynamical Systems

Stommel's Model

Vague Analogy to Atlantic Overturning Circulation

Gulf Stream reversed.

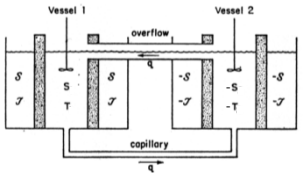
Gulf Stream flowing North



Stommel, TELLUS **XII** (1961)

Dynamical Systems

Stommel's Model



$y = \frac{T}{T^*} \quad x = \frac{S}{S^*} \quad d\tau = cdt$

$\delta = \frac{d}{c} \quad R = \frac{\beta S^*}{\alpha T^*} \quad \lambda = \frac{c}{4\rho_0 \alpha T^*} k$

flow rate $f = \frac{2q}{c}$

flow resistance

Let's increase the resistance in the capillary, so that it is harder for the water to flow between the vessels.

salinity $\frac{dx}{d\tau} = \delta(1-x) - |f|x$

temperature $\frac{dy}{d\tau} = 1 - y - |f|y$

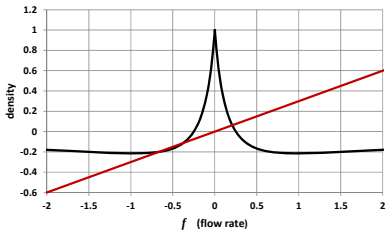
$\lambda f = -y + Rx$

Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 0.3$

Increase the flow resistance.



Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 0.3$

Increase the flow resistance.

stable manifold
unstable manifold
stable spiral
saddle
stable node

a
b
c

temperature
salinity
0
1

Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 0.3$

Increase the flow resistance.

Not much different, but it is easier to get to ***c***.

stable manifold
unstable manifold
stable spiral
saddle
stable node

a
b
c

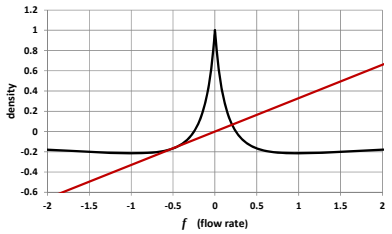
temperature
salinity
0
1

Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 0.33$

Increase the flow resistance. The saddle and the stable node start to merge.



Dynamical Systems

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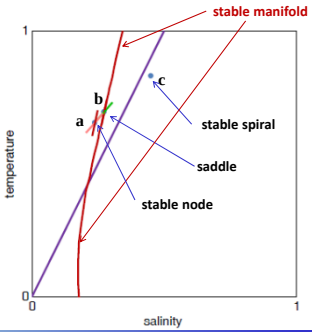

Dynamical Systems

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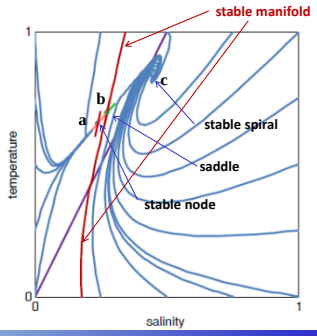
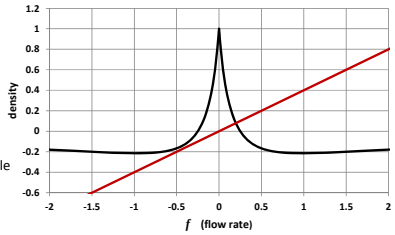


Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 0.4$

Increase the flow resistance. The saddle and the stable node have disappeared. The Gulf Stream will eventually reverse.



Dynamical Systems

Stommel's Model

$\delta = 1/6$
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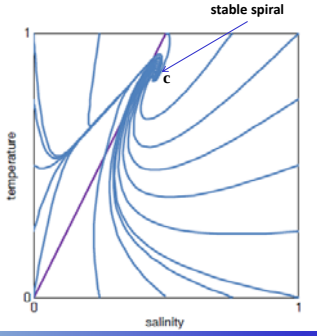
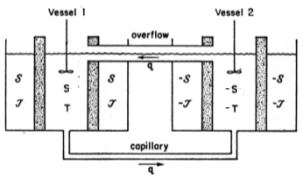
Increase the flow resistance. The saddle and the stable node have disappeared. The Gulf Stream will eventually reverse.



Dynamical Systems

Stommel's Model



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flow rate $f = \frac{2q}{c}$

flow resistance

Now decrease the resistance in the capillary, so that it is easier for the water to flow between the vessels.

salinity $\frac{dx}{d\tau} = \delta(1-x) - |f|x$

temperature $\frac{dy}{d\tau} = 1 - y - |f|y$

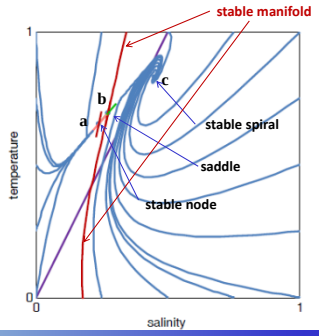
$\lambda f = -y + Rx$

Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 0.33$

The saddle and the stable node have re-emerged, but it is difficult to get to **a**. The Gulf Stream is still reversed.

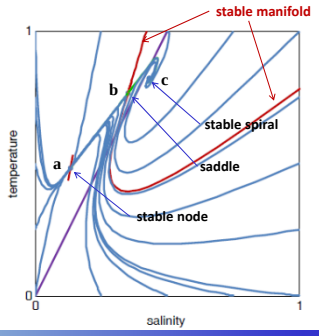


Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$

We are back to our original parameters, but the Gulf Stream is still reversed.

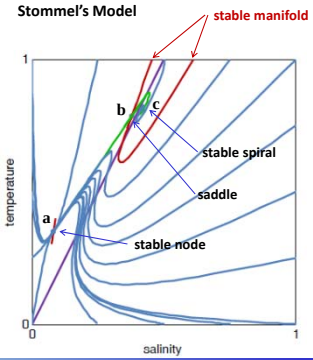


Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 0.1$

The flow resistance is below the original value. Point **a** is the dominant attractor. Perhaps the Gulf Stream will find a way to return to normal.

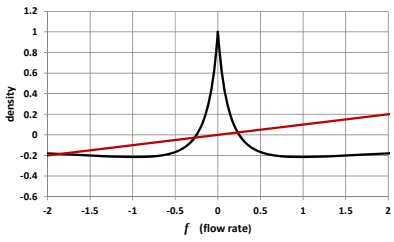


Dynamical Systems

Stommel's Model

$\delta = 1/6$
$R = 2$
$\lambda = 0.1$

The flow resistance is below the original value. Point **a** is the dominant attractor. Perhaps the Gulf Stream will find a way to return to normal.



Dynamical Systems

Stommel's Model

Henry Stommel, *Thermohaline Convection with Two Stable Regimes of Flow*, TELLUS **XII** (1961), 224-230.