Math 5490

October 22, 2014

Topics in Applied Mathematics: Introduction to the Mathematics of Climate

Mondays and Wednesdays 2:30 – 3:45

http://www.math.umn.edu/~mcgehee/teaching/Math5490-2014-2Fall/

Streaming video is available at
http://www.ima.umn.edu/videos/
Click on the link: "Live Streaming from 305 Lind Hall".

Participation:
https://umconnect.umn.edu/mathclimate

MCRN Mathematics and Climate Research Network
IMA Institute for Mathematics and Its Applications

Heat Imbalance

Monday's mistakes corrected.

Heat Imbalance

Table S1. Planetary Heat Storage: Ocean, Ice, Air and Land.

Energy required to melt ice and warm the air, land and ocean by specified amounts.[1]

Ocean warming by 1°C through 1 km depth of ocean. Heat storage is 1°C × 10⁵ g/cm² × 1 cal/g × 4.19 joules/cal × area Earth × 0.7 ~ 15 × 10²³ joules ~ **93 W yr/m²**.

Ice sheet melting to raise sea level 1 meter. Assume ice starts at −10°C and ends at mean ocean surface temperature (+15°C). Energy required is 100 cal/g (80 cal/g for melting). Energy for 1 meter of sea level: 100g/cm² × 100cal/g × 4.19 joules/cal × area Earth × 0.7 ~1.5 × 10²³ joules ~ **9.3 W yr/m²**.

Sea ice melting (all sea ice on planet). Assume ice starts at −10°C and ends at mean ocean surface temperature (+15°C), and that sea ice covers 4% of the planet with mean thickness 2.5 m. Energy required is 250 g cm² × 100 cal/g (80 cal/g for melting) × 4.19 joules/cal × 0.04 × area Earth ~ 2.14 × 10²² joules ~ **1.3 W yr/m²**.

Air warming by 1°C. The Earth's atmospheric mass is ~ 10 m of water. Heat capacity of air ~ 0.24 cal/g/°C. Energy to raise air temperature 1°C: 1°C × 1000 g/cm² × 0.24 cal/g/°C × 4.19 joules/cal × area Earth ~ 0.26 ×10²² joules ~ **0.32 W yr/m²**.

Land surface warming by 1°C. The depth of penetration of a thermal wave into the Earth's crust in 10 years, weighted by ΔT, is ~10 m. With density ~ 3 g/cm³, heat capacity ~ 0.2 cal/g/°C, and 0.29 fractional land coverage, land heat storage is 10³ cm ×3 g/cm³ × 0.2 cal/g/°C × 1°C ×4.19 joules/cal × area Earth × 0.29 ~ 0.37 × 10²² joules ~ **0.23 W yr**. [In a century the depth of penetration is ~3 times more than in a decade, so heat storage in a century due to 1°C warming is ~ **0.7 W yr/m²**.]

[1] Note that 1 W sec = 1 joule, # sec/year ~ π×10⁷, area Earth ~ 5.1×10¹⁸ cm², 1 W yr over full Earth ~ 1.61×10²² joules, ocean fraction of Earth ~ 0.7, 1 calorie ~ 4.19 joules.

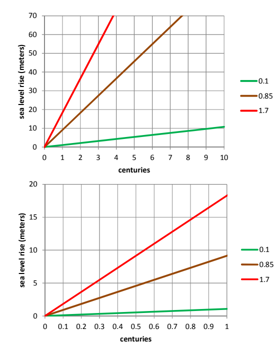
James Hansen, et al, *Earth's Energy Imbalance: Confirmation and Implications*, SCIENCE 308 (2005), p. 1431

Heat Imbalance

Suppose that all the heat imbalance went to melting the glaciers.

It takes 9.3 Wyr/m² to turn glaciers into 1 meter of ocean. If the heat imbalance is w W/m², the sea level would rise at the rate of $w/9.3$ meters per year. At the current imbalance of 0.85 W/m², the rate is about 0.091 meters per year, or 9.1 meters per century.

Melting all the glaciers would cause a sea level rise of about 70 meters and would take about 760 years at the current imbalance.

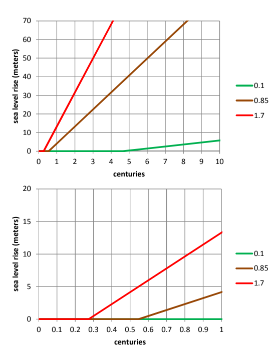


Heat Imbalance

Suppose now that all the heat imbalance first goes to raising the top kilometer of ocean by 0.5 °C, and then goes to melting the glaciers.

It takes 46.5 Wyr/m² to raise the temperature of a kilometer of ocean by 0.5 °C. If the heat imbalance is w W/m², the increase would be achieved in $46.5/w$ years, after which the sea level would rise at $w/9.3$ meters per year.

At the current imbalance of 0.85 W/m², the ocean temperature increase would delay the sea level rise by about 56 years.

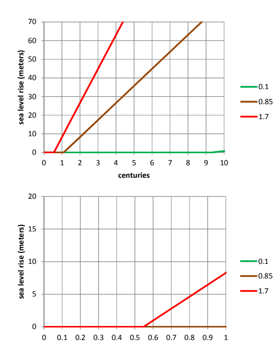


Heat Imbalance

Suppose instead that all the heat imbalance first goes to raising the top kilometer of ocean by 1 °C, and then goes to melting the glaciers.

It takes 93 Wyr/m² to raise the temperature of a kilometer of ocean by 1 °C. If the heat imbalance is w W/m², the increase would be achieved in $93/w$ years, after which the sea level would rise at $w/9.3$ meters per year.

At the current imbalance of 0.85 W/m², the ocean temperature increase would delay the sea level rise by about 112 years.



Heat Imbalance

Summary

Currently, it appears that the heat imbalance is mostly going to heating the ocean, not to melting ice. If this pattern continues, the danger for this century is more likely to come from weather changes than from sea level rise.

The current heat imbalance has the potential to raise the sea level by almost a meter per decade, a major threat to coastal cities worldwide.

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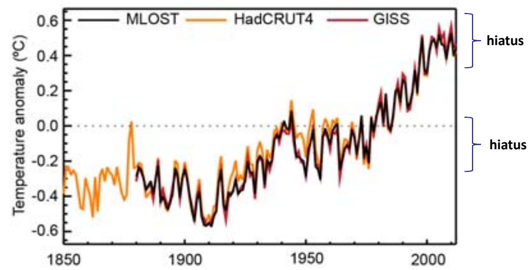
Heat Imbalance

What's Happening Now?



IPCC AR5 (2013) Figure 2.20

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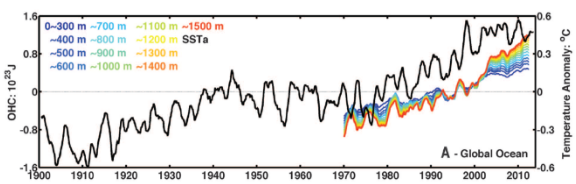
Heat Imbalance

What's Happening Now?

The heat imbalance is being absorbed by the ocean (at 1000 meters, not the surface).



Chou & Tung, Science 345 (2014) p 897

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Heat Imbalance

What's Happening Now?

The Good News

The surface temperatures are remaining fairly constant, so the perceived warming is small. This hiatus gives us an opportunity to address the basic problem (mitigation).

The Bad News

The surface temperatures are remaining fairly constant, so the perceived warming is small. This hiatus could lull us into complacency so that we do not address the basic problem.

There is evidence that hiatuses (hiati?) correspond to 60 year cycles of the AMOC. Will we experience another strong warming period in 30 years?

Chou & Tung, Science 345 (2014) p 897

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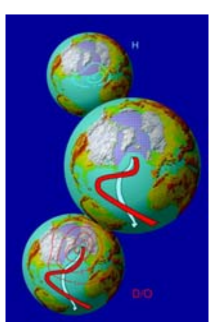
Ocean Circulation

Atlantic Meridional Overturning Circulation (AMOC)

Cold salty water is dense, so it descends in the North Atlantic. The rate of circulation is a function of the temperature and salinity and can change over time.



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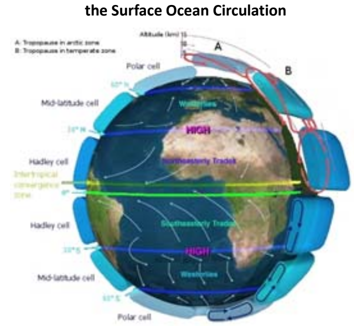
Ocean Circulation

Atmospheric Circulation Drives the Surface Ocean Circulation



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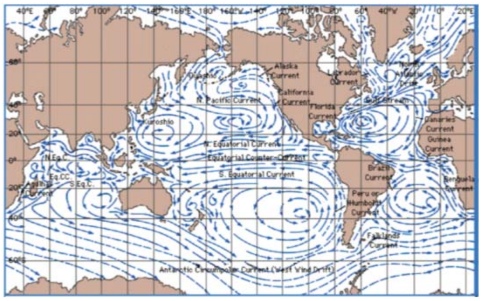
Ocean Circulation

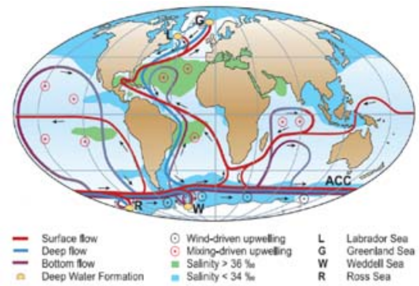
Surface Ocean Circulation



Kaper & Engler

Ocean Circulation

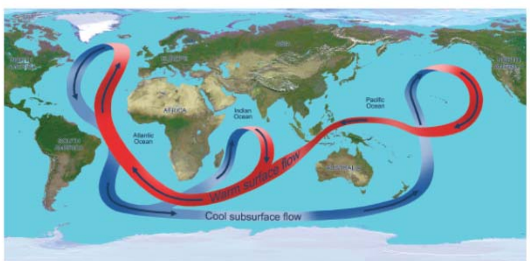
Overturning Ocean Circulation



Kaper & Engler

Ocean Circulation

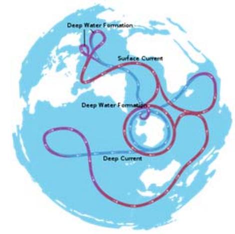
Simplified Overturning Ocean Circulation
“Conveyer Belt”, “Thermohaline Circulation.



Kaper & Engler

Ocean Circulation

Simplified Overturning Ocean Circulation
View Looking Down on Antarctic



http://commons.wikimedia.org/wiki/File:Conveyor_belt.svg

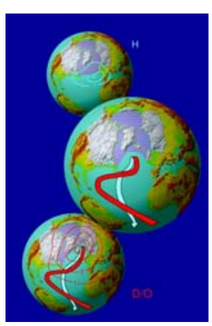
Ocean Circulation

Atlantic Meridional Overturning Circulation (AMOC)

Part of the “thermohaline circulation”.

thermo: heat
haline: salt

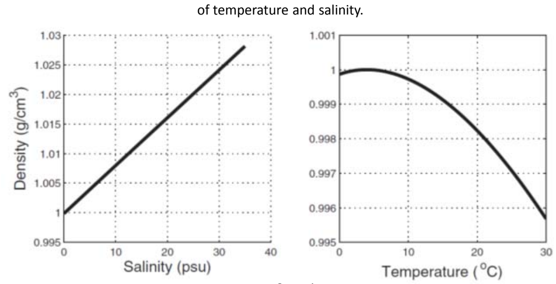
Cold salty water is dense, so it descends in the North Atlantic. The rate of circulation is a function of the temperature and salinity and can change over time.



Ocean Box Models

Thermohaline Circulation

The circulation is driven by density differences, which are functions of temperature and salinity.

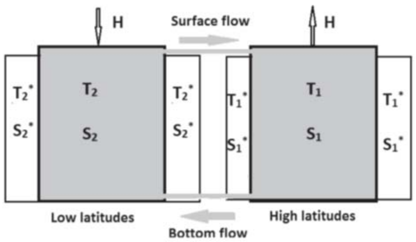


Kaper & Engler

Ocean Box Models

Stommel Model

Henry Stommel, *Thermohaline Convection with Two Stable Regimes of Flow*, TELLUS **XII** (1961), 224-230.



Kaper & Engler

Ocean Box Models

Stommel Model

$$\frac{dT}{dt} = c(T^* - T)$$
$$\frac{dS}{dt} = d(S^* - S)$$

T: temperature
S: salinity

Stars indicate constant bath temperature and salinity.

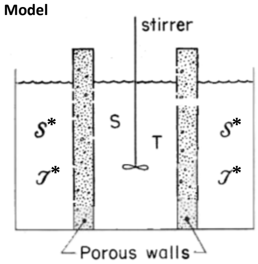


Fig. 1. The idealized experiment, consisting of a well stirred vessel of water with temperature T and salinity S (in general variable in time) separated by porous walls from and outside vessel whose temperature $\mathcal{T}$ and salinity $\mathcal{S}$ are maintained at constant values.

Stommel, TELLUS **XII** (1961)

Ocean Box Models

Dynamical Systems Approach

$$\frac{dT}{dt} = c(T^* - T)$$
$$\frac{dS}{dt} = d(S^* - S)$$

Both equation have the form:

$$\frac{dx}{dt} = \alpha(x^* - x), \text{ where } \alpha \text{ and } x^* \text{ are constant.}$$

Consider first

$$\frac{dx}{dt} = -\alpha x$$

General solution

$$x(t) = ce^{-\alpha t}, \text{ where } c \text{ is an arbitrary constant.}$$

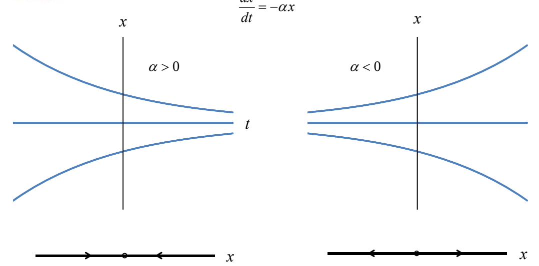
Let $x_0 = x(0) = c$

$$\boxed{x(t) = x_0 e^{-\alpha t}}$$

Ocean Box Models

Dynamical Systems Approach

$$\frac{dx}{dt} = -\alpha x$$



Ocean Box Models

Dynamical Systems Approach

Back to:

$$\frac{dx}{dt} = \alpha(x^* - x), \text{ where } \alpha \text{ and } x^* \text{ are constant.}$$

Equilibrium solution

$$0 = \frac{dx}{dt} = \alpha(x^* - x), \text{ so } x(t) = x^* \text{ (constant).}$$

General solution

$$x(t) = x^* + ce^{-\alpha t}, \text{ where is an arbitrary constant.}$$

$$x_0 = x(0) = x^* + c, \text{ so } c = x_0 - x^*.$$

$$\boxed{x(t) = x^* + (x_0 - x^*)e^{-\alpha t}}$$

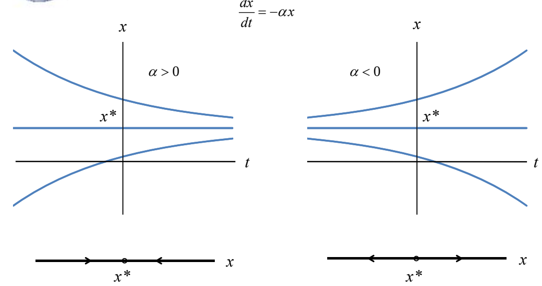
"Asymptotic stability" of the equilibrium solution →

If $\alpha > 0$, $x(t) \to x^*$ as $t \to \infty$.

Ocean Box Models

Dynamical Systems Approach

$$\frac{dx}{dt} = -\alpha x$$



stable unstable

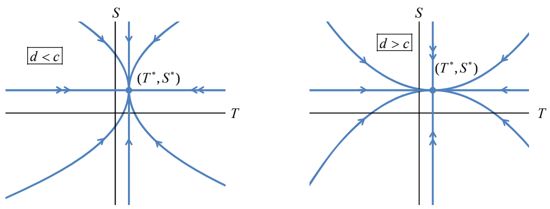
Ocean Box Models

Dynamical Systems Approach

$$\frac{dT}{dt} = c(T^* - T)$$
$$\frac{dS}{dt} = d(S^* - S)$$

Solve each separately:

$$T(t) = T^* + (T_0 - T^*)e^{-ct}$$
$$S(t) = S^* + (S_0 - S^*)e^{-dt}$$



"Phase Portraits"

Ocean Box Models

Dynamical Systems Approach

$$\frac{dT}{dt} = c(T^* - T)$$
$$\frac{dS}{dt} = d(S^* - S)$$

"Non-dimensionalize"

We want to (1) take away as many units as possible, and (2) get rid of as many constants as possible.

Let $y = \frac{T}{T^*}$, $x = \frac{S}{S^*}$

$$\frac{dy}{dt} = \frac{1}{T^*}\frac{dT}{dt} = \frac{c}{T^*}(T^* - T) = c\left(1 - \frac{T}{T^*}\right) = c(1-y)$$
$$\frac{dx}{dt} = \frac{1}{S^*}\frac{dS}{dt} = \frac{d}{S^*}(S^* - S) = d\left(1 - \frac{S}{S^*}\right) = d(1-x)$$

$$\boxed{\begin{aligned}\frac{dy}{dt} &= c(1-y)\\ \frac{dx}{dt} &= d(1-x)\end{aligned}}$$

The new variables x and y are "dimensionless".

Ocean Box Models

Dynamical Systems Approach

$$\frac{dy}{dt} = c(1-y)$$
$$\frac{dx}{dt} = d(1-x)$$

"Time scaling"

We are not wedded to seconds. Introduce a new time variable τ :

$$d\tau = c\,dt$$

$$\frac{dy}{d\tau} = \frac{dy}{c\,dt} = \frac{1}{c}\frac{dy}{dt} = \frac{1}{c}c(1-y) = 1-y$$
$$\frac{dx}{d\tau} = \frac{dx}{c\,dt} = \frac{1}{c}\frac{dx}{dt} = \frac{1}{c}d(1-x) = \frac{d}{c}(1-x)$$

$$\boxed{\begin{aligned}\frac{dy}{d\tau} &= 1-y\\ \frac{dx}{d\tau} &= \frac{d}{c}(1-x)\end{aligned}}$$

Ocean Box Models

Dynamical Systems Approach

$$\frac{dy}{d\tau} = 1-y$$
$$\frac{dx}{d\tau} = \frac{d}{c}(1-x)$$

Finally, introduce the ratio of the rates at which the temperature and salinity equilibrate.

$$\delta = \frac{d}{c}$$

$$\begin{aligned}\frac{dT}{dt} &= c(T^* - T)\\ \frac{dS}{dt} &= d(S^* - S)\end{aligned} \quad \Longrightarrow \quad \boxed{\begin{aligned}\frac{dy}{d\tau} &= 1-y\\ \frac{dx}{d\tau} &= \delta(1-x)\end{aligned}}$$

By rescaling the temperature, salinity, and time, we have reduced the original system with four parameters, to one with just one parameter.