Math 5490

November 10, 2014

Topics in Applied Mathematics: Introduction to the Mathematics of Climate

Mondays and Wednesdays 2:30 – 3:45

http://www.math.umn.edu/~mcgehee/teaching/Math5490-2014-2Fall/

Streaming video is available at
http://www.ima.umn.edu/videos/
Click on the link: "Live Streaming from 305 Lind Hall".

Participation:
https://umconnect.umn.edu/mathclimate

MCRN Mathematics and Climate Research Network
IMA Institute for Mathematics and Its Applications

Dynamical Systems

Nonlinear Systems

$\frac{dx}{dt} = f(x)$ Rest point p: $f(p) = 0$

Linear approximation:
$f(x) \approx f(p) + Df(p)(x-p) = Df(p)(x-p)$

Introduce $\xi = x - p$.
Then $f(x) = f(p+\xi) \approx Df(p)\xi$

$$\frac{d\xi}{dt} = \frac{dx}{dt} = f(x) = f(p+\xi) \approx Df(p)\xi$$

Basic Idea

If ξ is small, i.e., if x is close to p, then solutions of $\frac{d\xi}{dt} = f(\xi + p)$ are close to solutions of $\frac{d\xi}{dt} = Df(p)\xi$.

In particular, the rest point p is asymptotically stable for $\frac{dx}{dt} = f(x)$ if the origin is asymptotically stable for $\frac{d\xi}{dt} = Df(p)\xi$.

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Dynamical Systems

Flows

"Vector Fields Determine Flows"

A *flow* is a continuous map $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfying
$\varphi(x,0) = x$, for all x,
$\varphi(x, t+s) = \varphi(\varphi(x,t), s)$, for all x, t, and s.

Alternate Notation $\varphi^t(x) = \varphi(x,t)$

$\varphi^0 = \mathrm{id}$

Group Property → $\varphi^{t+s} = \varphi^t \circ \varphi^s$

Smooth: C^n, $n > 0$
(n times continuously differentiable)

Big Theorem

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, then the initial value problem
$\dot{x} = f(x)$, $x(0) = x_0$,
defines a flow $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfying
$\dot{\varphi}(x_0, t) = f(\varphi(x_0, t))$, $\varphi(x_0, 0) = x_0$.
Also, φ is a smooth as f.

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Dynamical Systems

Flows

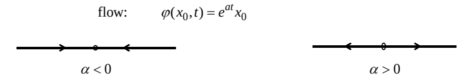
"Vector Fields Determine Flows"

Example

vector field: $\dot{x} = ax$, $x \in \mathbb{R}$, initial value: $x(0) = x_0$

solution: $x = e^{at}x_0$

flow: $\varphi(x_0, t) = e^{at}x_0$



Check properties:

Group Property

$\varphi(x_0, 0) = e^{a \cdot 0}x_0 = x_0$

$\varphi(x_0, t+s) = e^{a(t+s)}x_0 = e^{at}e^{as}x_0 = e^{at}\varphi(x_0, s) = \varphi(\varphi(x_0, s), t)$

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Dynamical Systems

Flows

"Vector Fields Determine Flows"

Example

vector field: $\dot{x} = Ax$, $x \in \mathbb{R}^n$ initial value: $x(0) = x_0$

solution: $x = e^{tA}x_0$

flow: $\varphi(x_0, t) = e^{tA}x_0$

Check properties:

Group Property

$\varphi(x_0, 0) = e^{0 \cdot A}x_0 = x_0$

$\varphi(x_0, t+s) = e^{(t+s)A}x_0 = e^{tA}e^{sA}x_0 = e^{tA}\varphi(x_0, s) = \varphi(\varphi(x_0, s), t)$

Experts Only

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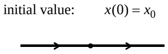
Dynamical Systems

Flows

"Vector Fields Determine Flows"

Example

vector field: $\dot{x} = x^2$, $x \in \mathbb{R}$
initial value: $x(0) = x_0$



Calculus

$$\frac{dx}{dt} = x^2 \to x^{-2}dx = dt \to \int_{x_0}^{x} x^{-2}dx = \int_0^t dt \to -x^{-1}\Big|_{x_0}^{x} = t$$

$$-\frac{1}{x} + \frac{1}{x_0} = t \to \frac{1}{x} = \frac{1}{x_0} - t = \frac{1 - x_0 t}{x_0} \to x = \frac{x_0}{1 - x_0 t}$$

solution: $x = \frac{x_0}{1 - x_0 t}$

flow: $\varphi(x_0, t) = \frac{x_0}{1 - x_0 t}$

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Dynamical Systems
Flows
"Vector Fields Determine Flows"

Example

vector field: $\dot{x} = x^2, \quad x \in \mathbb{R}$ initial value: $x(0) = x_0$

flow: $\varphi(x_0, t) = \dfrac{x_0}{1 - x_0 t}$

Check properties:

$$\varphi(x_0, 0) = \frac{x_0}{1 - x_0 \cdot 0} = x_0$$

group property

$$\varphi(\varphi(x_0,t),s) = \frac{\varphi(x_0,t)}{1-\varphi(x_0,t)s} = \frac{\dfrac{x_0}{1-x_0 t}}{1-\dfrac{x_0}{1-x_0 t}s} = \frac{x_0}{1-x_0 t - x_0 s} = \frac{x_0}{1-x_0(t+s)} = \varphi(x_0, t+s)$$

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Dynamical Systems
Flows
"Vector Fields Determine Flows"

Calculus

Example

vector field: $\dot{x} = x^2, \quad x \in \mathbb{R}$ initial value: $x(0) = x_0$

flow: $\varphi(x_0, t) = \dfrac{x_0}{1 - x_0 t}$

Issue:

Solutions do not exist for all time.

$$\varphi(x_0, t) = \frac{x_0}{1 - x_0 t} \to \infty \ \text{ as } \ t \to \frac{1}{x_0}$$

local flow: Solutions exist for some time interval.

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Dynamical Systems
Flows
"Vector Fields Determine Flows"

Example

vector field: $\dot{x} = 1 - x^2, \quad x \in \mathbb{R}$

initial value: $x(0) = x_0$

Calculus

$$\frac{dx}{dt} = 1 - x^2 \ \to \ (1-x^2)^{-1}dx = dt \ \to \ \int_{x_0}^{x} (1-x^2)^{-1}dx = \int_0^t dt \ \to \ \tanh^{-1} x \Big|_{x_0}^{x} = t$$

$$\tanh^{-1} x - \tanh^{-1} x_0 = t \ \to \ \tanh^{-1} x = \tanh^{-1} x_0 + t \ \to \ x = \tanh\left(\tanh^{-1} x_0 + t\right)$$

$$x = \frac{\tanh(\tanh^{-1} x_0) + \tanh t}{1 + \tanh(\tanh^{-1} x_0)\tanh t} = \frac{x_0 + \tanh t}{1 + x_0 \tanh t}$$

flow: $\varphi(x_0, t) == \dfrac{x_0 + \tanh t}{1 + x_0 \tanh t}$

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Dynamical Systems
Flows
"Vector Fields Determine Flows"

Example

vector field: $\dot{x} = 1 - x^2, \quad x \in \mathbb{R}$ initial value: $x(0) = x_0$

flow: $\varphi(x_0, t) = \dfrac{x_0 + \tanh t}{1 + x_0 \tanh t}$

Group Property

$$\varphi(x_0, t+s) = \frac{x_0 + \tanh(t+s)}{1 + x_0 \tanh(t+s)} = \frac{x_0 + \dfrac{\tanh t + \tanh s}{1 + \tanh t \tanh s}}{1 + x_0 \dfrac{\tanh t + \tanh s}{1 + \tanh t \tanh s}} = \frac{x_0 + x_0 \tanh t \tanh s + \tanh t + \tanh s}{1 + \tanh t \tanh s + x_0(\tanh t + \tanh s)}$$

equal

$$\varphi(\varphi(x_0,t),s) = \frac{\varphi(x_0,t) + \tanh t}{1 + \varphi(x_0,t)\tanh t} = \frac{\dfrac{x_0 + \tanh t}{1 + x_0 \tanh t} + \tanh s}{1 + \dfrac{x_0 + \tanh t}{1 + x_0 \tanh t}\tanh s} = \frac{x_0 + \tanh t + \tanh s + x_0 \tanh t \tanh s}{1 + x_0 \tanh t + x_0 \tanh s + \tanh t \tanh s}$$

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Dynamical Systems
Flows
"Vector Fields Determine Flows"

Example

vector field: $\dot{x} = 1 - x^2, \quad x \in \mathbb{R}$

local flow: $\varphi(x_0, t) = \dfrac{x_0 + \tanh t}{1 + x_0 \tanh t}$

Issue:

Solutions do not exist for all time.

$$x_0 < -1 \ \Rightarrow \ \varphi(x_0, t) = \frac{x_0 + \tanh t}{1 + x_0 \tanh t} \to -\infty \ \text{ as } \ t \to \tanh^{-1}(-1/x_0)$$

$$x_0 > +1 \ \Rightarrow \ \varphi(x_0, t) = \frac{x_0 + \tanh t}{1 + x_0 \tanh t} \to +\infty \ \text{ as } \ t \to \tanh^{-1}(-1/x_0)$$

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Dynamical Systems
Flows

Backwards Time

vector field: $\dot{x} = f(x), \quad x(0) = x_0, \quad x \in \mathbb{R}^n$

flow: $\dot{\varphi}(x_0, t) = f\left(\varphi(x_0, t)\right)$

Let $\psi(x_0, t) = \varphi(x_0, -t)$

then $\dfrac{\partial}{\partial t}\psi(x_0, t) = \dfrac{\partial}{\partial t}\varphi(x_0, -t) = -\dot{\varphi}(x_0, -t) = -f\left(\varphi(x_0, -t)\right) = -f\left(\psi(x_0, t)\right)$

So $\psi(x_0, t)$ satisfies

$\dot{x} = -f(x), \quad x(0) = x_0$

Going backward in time is the same as following the negative of the vector field.

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Dynamical Systems

Flows

"Vector Fields Determine Flows"

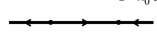
Example

vector field: $\dot{x} = 1 - x^2, \quad x \in \mathbb{R}$

local flow: $\varphi(x_0, t) = \dfrac{x_0 + \tanh t}{1 + x_0 \tanh t}$

vector field: $\dot{x} = -\left(1 - x^2\right), \quad x \in \mathbb{R}$

local flow: $\psi(x_0, t) = \dfrac{x_0 - \tanh t}{1 - x_0 \tanh t}$



$$\psi(x_0, t) = \frac{x_0 - \tanh t}{1 - x_0 \tanh t} = \frac{x_0 + \tanh(-t)}{1 + x_0 \tanh(-t)} = \varphi(x_0, -t)$$

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Dynamical Systems

Nonlinear Systems

Stability

$\dfrac{dx}{dt} = f(x)$ Rest point $p: \ f(p) = 0$

The rest point p is called ***Lyapunov stable*** (or simply ***stable***) if,
for all $\varepsilon > 0$, there exists a $\delta > 0$ such that
$|x_0 - p| < \delta \ \Rightarrow \ |\varphi(x_0, t) - p| < \varepsilon$ for all $t > 0$.
It is called ***unstable*** if it is not stable.

The rest point p is called ***asymptotically stable*** if it is stable and if
there exists a $\delta > 0$ such that
$|x_0 - p| < \delta \ \Rightarrow \ \varphi(x_0, t) \to p$ as $t \to \infty$.

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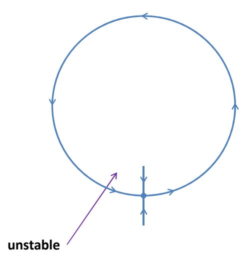
Dynamical Systems

Nonlinear Systems

Stability



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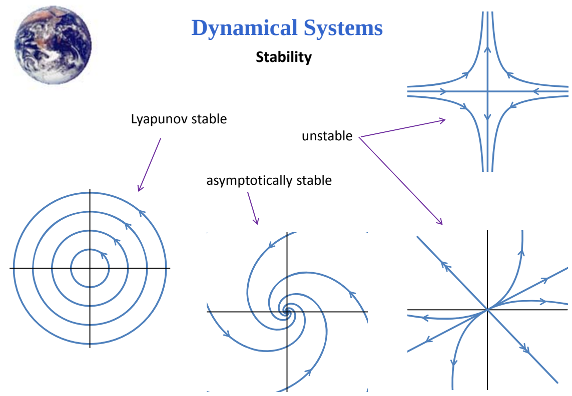
Dynamical Systems

Stability



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Dynamical Systems

Nonlinear Systems

$\dfrac{dx}{dt} = f(x)$ Rest point $p: \ f(p) = 0$

variational equation near p: $\dfrac{d\xi}{dt} = Df(p)\xi$ ← $\xi \approx x - p$

↖ Jacobian matrix

If all of the eigenvalues of the Jacobian matrix have negative real part, then the rest point p is asymptotically stable.

If any eigenvalue of the Jacobian matrix has positive real part, then the rest point p is unstable.

If all of the eigenvalues of the Jacobian matrix have positive real part, then the rest point p is asymptotically stable for the backwards flow.

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Dynamical Systems

Nonlinear Systems

$\dfrac{dx}{dt} = f(x), \quad x \in \mathbb{R}^2$ Rest point $p: \ f(p) = 0$

What about saddles?

If one of the eigenvalues of the Jacobian matrix is positive and the other is negative, then there are two smooth curves intersecting at p, $W^s(p)$ (the stable manifold), and $W^u(p)$ (the unstable manifold) satisfying these properities:

$x \in W^s(p) \ \Leftrightarrow \ \varphi(x,t) \to p$ as $t \to \infty$

$x \in W^u(p) \ \Leftrightarrow \ \varphi(x,t) \to p$ as $t \to -\infty$

$W^s(p)$ is tangent at p to the eigenvector corresponding to the negative eigenvalue

$W^u(p)$ is tangent at p to the eigenvector corresponding to the positive eigenvalue

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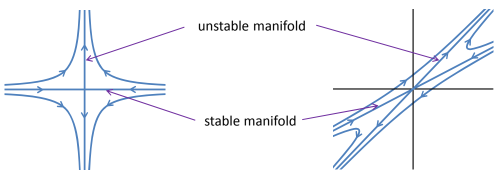
Nonlinear Systems

$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^2$ Rest point $p: \quad f(p) = 0$

What about saddles?



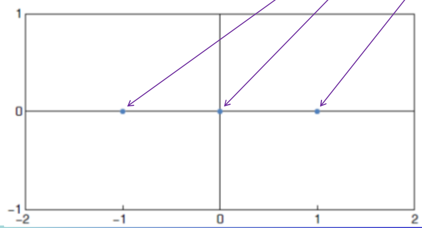
Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

rest points: $(-1,0)$, $(0,0)$, and $(1,0)$



Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

$$Df(x,y) = \begin{bmatrix} 1-3x^2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$Df(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

Rest points: $(-1,0)$, $(0,0)$, and $(1,0)$

$$Df(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

sinks

eigenvalues: $-2 \quad -1$

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

fast slow

saddle

eigenvalues: $1 \quad -1$

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

unstable stable

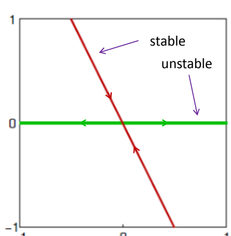
Dynamical Systems

Nonlinear Systems

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

Example

saddle $\dot{\xi} = A\xi, \quad A = Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$



eigenvalues: $1 \quad -1$

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

unstable stable

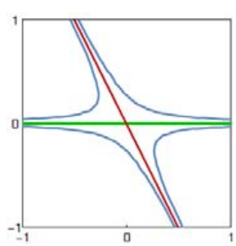
Dynamical Systems

Nonlinear Systems

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

Example

saddle $\dot{\xi} = Ax, \quad A = Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$



eigenvalues: $1 \quad -1$

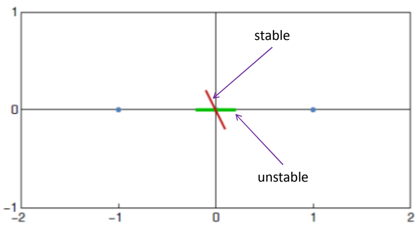
eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$



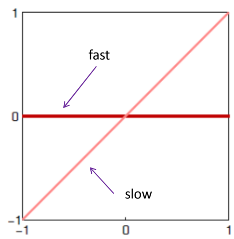
Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

stable node $\dot{\xi} = A\xi, \quad A = Df(-1,0) = Df(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$



eigenvalues: $-2 \quad -1$

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

fast slow

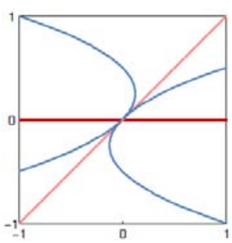
Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

stable node $\dot{\xi} = A\xi, \quad A = Df(-1,0) = Df(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$



eigenvalues: $-2 \quad -1$

eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
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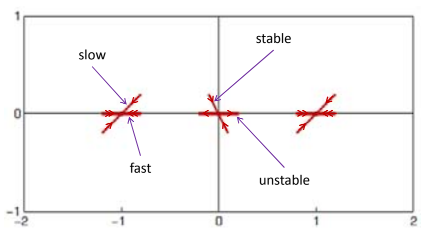

Dynamical Systems

Nonlinear Systems

Example

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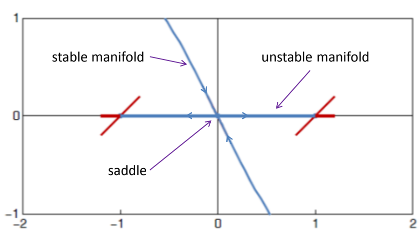
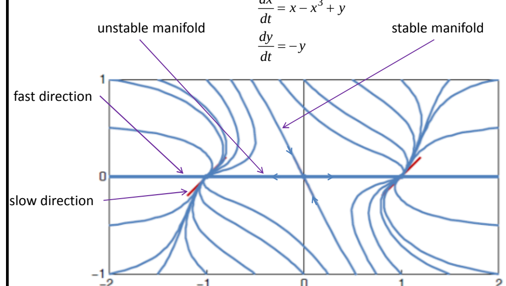


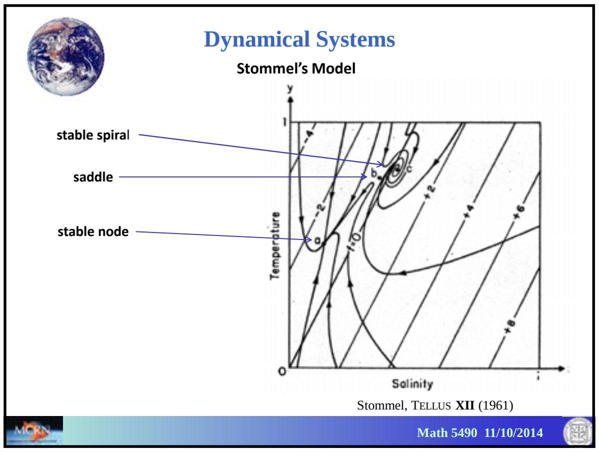
Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$




Dynamical Systems
Stommel's Model
stable spiral
saddle
stable node
Temperature
Salinity
Stommel, TELLUS XII (1961)
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