Math 5490
November 5, 2014

Topics in Applied Mathematics: Introduction to the Mathematics of Climate

Mondays and Wednesdays 2:30 – 3:45

http://www.math.umn.edu/~mcgehee/teaching/Math5490-2014-2Fall/

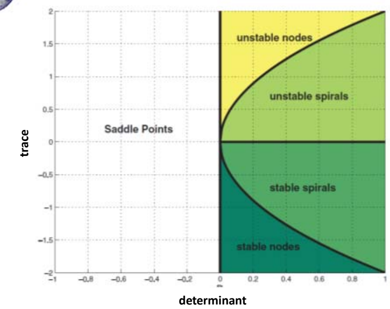
Streaming video is available at
http://www.ima.umn.edu/videos/
Click on the link: "Live Streaming from 305 Lind Hall".

Participation:
https://umconnect.umn.edu/mathclimate

MCRN Mathematics and Climate Research Network

IMA Institute for Mathematics and Its Applications

Dynamical Systems
Classification



Kaper & Engler, 2013

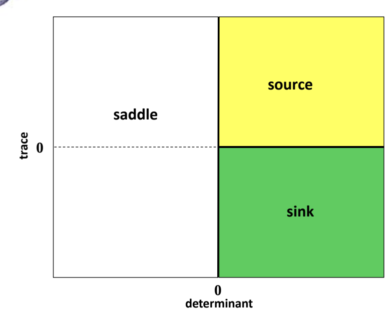
Dynamical Systems
Topological Classification

If neither eigenvalue has zero real part, then the system is called ***hyperbolic***, in which case, there are only three classes:

1. **saddles**: One positive eigenvalue and one negative. The determinant is negative.
2. **sources**: Both eigenvalues have positive real part. The determinant is positive, and the trace is positive.
3. **sinks**: Both eigenvalues have negative real part. The determinant is positive, and the trace is negative.

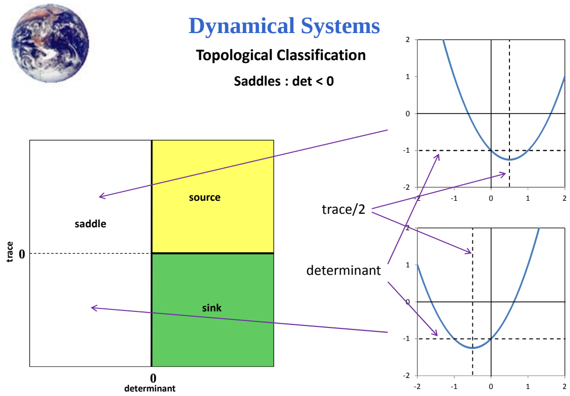
Every system in one of the three categories looks the same to a topologist (topological conjugacy).

Dynamical Systems
Topological Classification



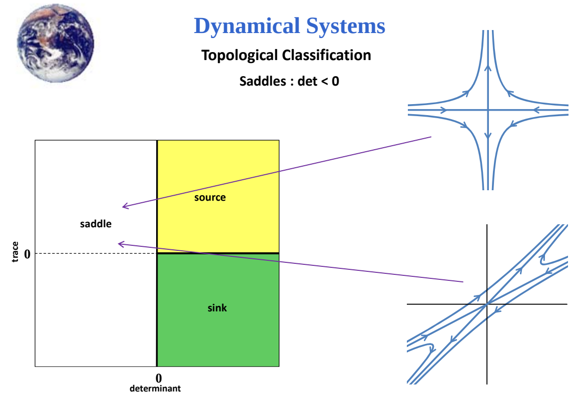
Dynamical Systems
Topological Classification

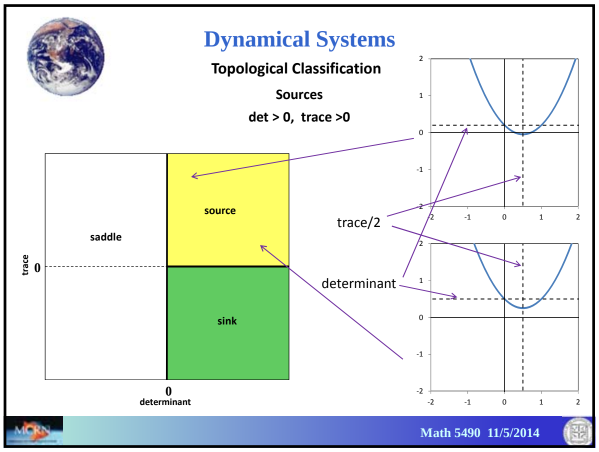
Saddles : det < 0



Dynamical Systems
Topological Classification

Saddles : det < 0




Dynamical Systems
Topological Classification
Sources
det > 0, trace >0
saddle
source
sink
trace
0
0
determinant
trace/2
determinant
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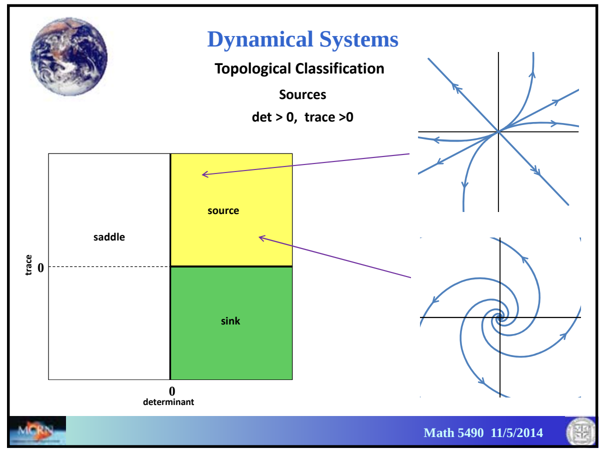
Dynamical Systems
Topological Classification
Sources
det > 0, trace >0
saddle
source
sink
trace
0
0
determinant
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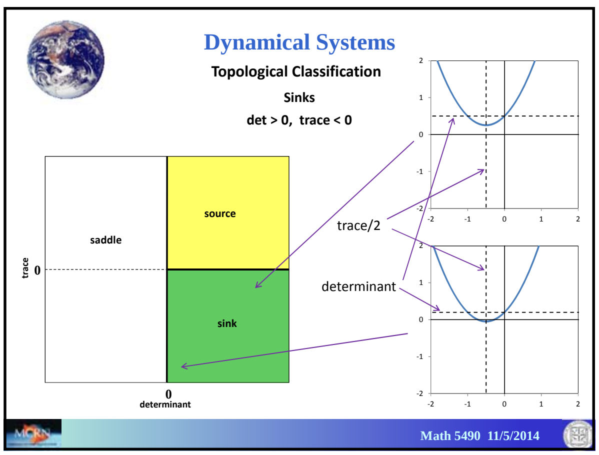
Dynamical Systems
Topological Classification
Sinks
det > 0, trace < 0
saddle
source
sink
trace
0
0
determinant
trace/2
determinant
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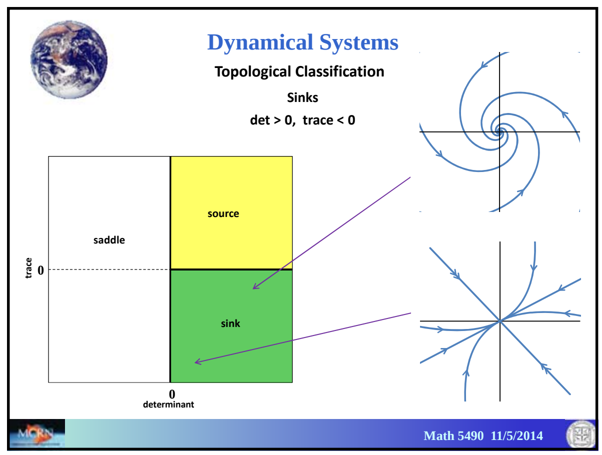
Dynamical Systems
Topological Classification
Sinks
det > 0, trace < 0
saddle
source
sink
trace
0
0
determinant
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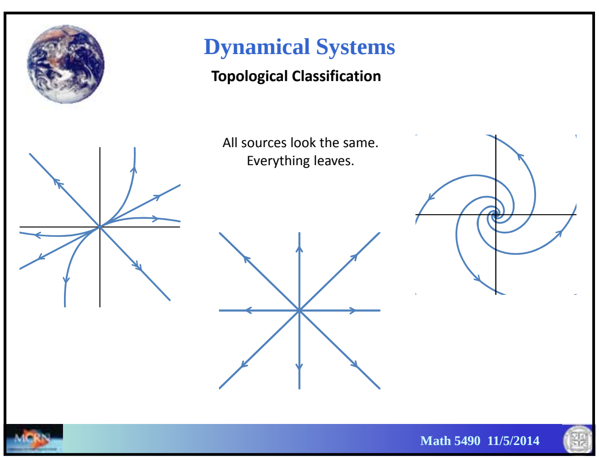
Dynamical Systems
Topological Classification
All sources look the same.
Everything leaves.
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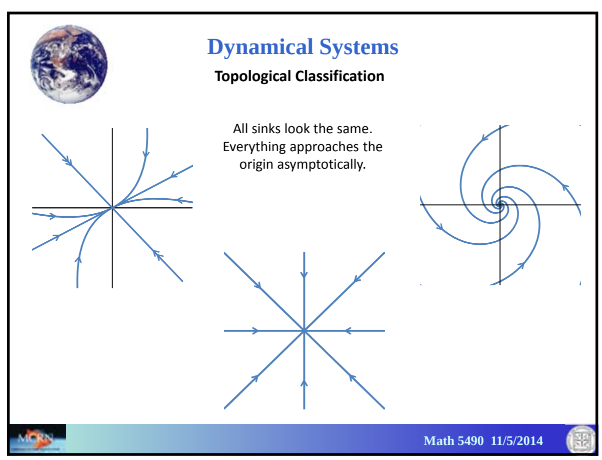
Dynamical Systems
Topological Classification
All sinks look the same.
Everything approaches the
origin asymptotically.
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Dynamical Systems

Topological Classification

One Variable

$$\frac{dx}{dt} = -\alpha x \quad \alpha > 0$$

$$\text{Let } x = |u|^{\alpha-1} u = \begin{cases} u^{\alpha} & u > 0 \\ -u^{\alpha} & u > 0 \end{cases}$$

$u > 0$

$$\frac{dx}{dt} = \alpha u^{\alpha-1} \frac{du}{dt} = -\alpha x = -\alpha u^{\alpha}$$

$$\boxed{\frac{du}{dt} = -u}$$

$u < 0$

$$\frac{dx}{dt} = -\alpha u^{\alpha-1} \frac{du}{dt} = -\alpha x = \alpha u^{\alpha}$$

$$\boxed{\frac{du}{dt} = -u}$$

Continuous,
but not smooth at 0.

$$\frac{dx}{dt} = -\alpha x \quad \alpha > 0 \Longrightarrow x = |u|^{\alpha-1} u \Longrightarrow \frac{du}{dt} = -u$$

Dynamical Systems

Topological Classification

Two Variables

Saddles

$$\frac{dx}{dt} = -\alpha x \quad \alpha > 0, \quad \frac{dy}{dt} = \beta x \quad \beta > 0 \Longrightarrow x = |u|^{\alpha-1} u, \quad y = |v|^{\beta-1} v \Longrightarrow \frac{du}{dt} = -u, \quad \frac{dv}{dt} = v$$

Continuous,
but not smooth at (0,0).

topologically conjugate

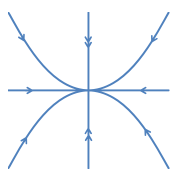
Dynamical Systems

Topological Classification

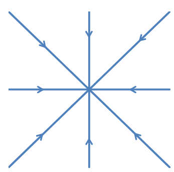
Sinks

$$\frac{dx}{dt} = -\alpha x \quad \alpha > 0, \quad \frac{dy}{dt} = -\beta x \quad \beta > 0 \Longrightarrow x = |u|^{\alpha-1} u, \quad y = |v|^{\beta-1} v \Longrightarrow \frac{du}{dt} = -u, \quad \frac{dv}{dt} = -v$$

Continuous,
but not smooth at (0,0).



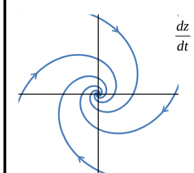
$\Longleftrightarrow$ topologically conjugate



Dynamical Systems

Topological Classification

What about spirals?

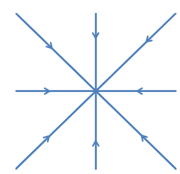


$$\frac{dz}{dt} = (-1-i)x$$

$$z = |w|^{i}\, w = e^{i \log|w|} w$$

Continuous,
but not smooth at 0.

$$\frac{dw}{dt} = -w$$



Dynamical Systems

Topological Classification

Computation

$$\boxed{\frac{dz}{dt} = (-1-i)z} \qquad z = |w|^{i}\, w = (w\bar{w})^{i/2} w = w^{(i/2+1)} \bar{w}^{i/2}$$

$$\dot{z} \equiv \frac{dz}{dt} = \left(\tfrac{i}{2}+1\right) w^{i/2} \dot{w} \bar{w}^{i/2} + w^{(i/2+1)}\, \tfrac{i}{2}\, \bar{w}^{(i/2-1)} \dot{\bar{w}}$$

$$= \left(\tfrac{i}{2}+1\right) |w|^{i}\, \dot{w} + \tfrac{i}{2} |w|^{i} \frac{w}{\bar{w}} \dot{\bar{w}}$$

$$= -(1+i)z = -(1+i)|w|^{i}\, w$$

Multiply by $\bar{w}/|w|^{i}$

$$\left(\tfrac{i}{2}+1\right) \bar{w}\dot{w} + \tfrac{i}{2} w\dot{\bar{w}} = -(1+i)\bar{w}w$$

complex conjugate

$$-\tfrac{i}{2} \bar{w}\dot{w} + \left(-\tfrac{i}{2}+1\right) w\dot{\bar{w}} = -(1-i)\bar{w}w$$

add

$$\bar{w}\dot{w} + w\dot{\bar{w}} = -2\bar{w}w$$

$$w\dot{\bar{w}} = -\bar{w}\dot{w} - 2\bar{w}w$$

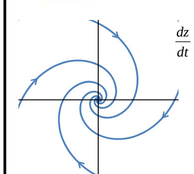
$$\left(\tfrac{i}{2}+1\right) \bar{w}\dot{w} + \tfrac{i}{2}\left(-\bar{w}\dot{w} - 2\bar{w}w\right) = -(1+i)\bar{w}w$$

$$\bar{w}\dot{w} - i\bar{w}w = -(1+i)\bar{w}w \longrightarrow \bar{w}\dot{w} - \bar{w}w \longrightarrow \boxed{\frac{dw}{dt} = -w}$$

Dynamical Systems

Topological Classification

What about spirals?



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Continuous,
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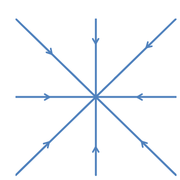
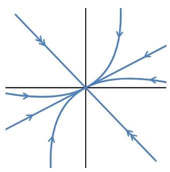
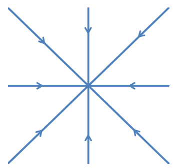
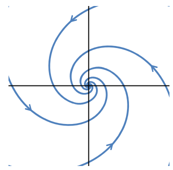


Dynamical Systems
Topological Classification

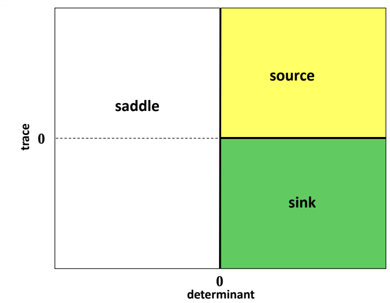


All sinks are topologically conjugate, as are all sources and all saddles.





Dynamical Systems
Topological Classification



Dynamical Systems
Nonlinear Systems
One Variable

$$\frac{dx}{dt} = f(x)$$

Rest points: $\frac{dx}{dt} = 0 \iff f(x) = 0$

If $f(p) = 0$, then $x(t) = p$ (constant) is a solution.

Example

$$\frac{dx}{dt} = x - x^3$$

Rest points: $x - x^3 = 0 \iff x(1-x)(1+x) = 0$

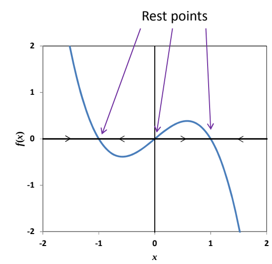
$-1,\ 0,\ \text{and}\ 1$

Dynamical Systems
Nonlinear Systems
One Variable

Example

$$\frac{dx}{dt} = x - x^3$$

What about the stability of the rest points?



Dynamical Systems
Nonlinear Systems

$\frac{dx}{dt} = f(x)$ Rest point p: $f(p) = 0$

Linear approximation:

$$f(x) \approx f(p) + f'(p)(x-p) = f'(p)(x-p)$$

Introduce $\xi = x - p$.

Then $f(x) = f(p+\xi) \approx f'(p)\xi$

$$\frac{d\xi}{dt} = \frac{dx}{dt} = f(x) = f(p+\xi) \approx f'(p)\xi$$

Basic Idea

If ξ is small, i.e., if x is close to p, then solutions of $\frac{d\xi}{dt} = f(\xi + p)$ are close to solutions of $\frac{d\xi}{dt} = f'(p)\xi$.

In particular, the rest point p is asymptotically stable for $\frac{dx}{dt} = f(x)$ if the origin is asymptotically stable for $\frac{d\xi}{dt} = f'(p)\xi$.

Dynamical Systems
Nonlinear Systems

$\frac{dx}{dt} = f(x)$ Rest point p: $f(p) = 0$

Criteria

The rest point p is asymptotically stable for $\frac{dx}{dt} = f(x)$ if $f'(p) < 0$.

It is unstable if $f'(p) > 0$.

Dynamical Systems

Nonlinear Systems

Example

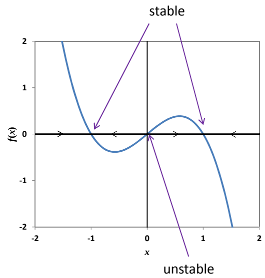
$$\frac{dx}{dt} = f(x) = x - x^3$$

Rest points: $-1,\ 0,$ and 1

$$f'(x) = 1 - 3x^2$$

$$f'(-1) = f'(1) = -2, \quad f'(0) = 1$$

Rest points -1 and 1 are stable, rest point 0 is unstable.



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Dynamical Systems

Nonlinear Systems

Two Variables

$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2)$$

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^2, \quad f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\frac{dx}{dt} = 0 \quad \Leftrightarrow \quad f(x) = 0$$

If $f(p) = 0$, then $x(t) = p$ (constant) is a solution (rest point)

$$f(p) = 0 \quad \Leftrightarrow \quad \begin{array}{l} f_1(p_1, p_2) = 0 \\ f_1(p_1, p_2) = 0 \end{array} \qquad \begin{array}{l} x_1(t) = p_1 \\ x_2(t) = p_2 \end{array}$$

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Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

Rest points:

$$\begin{array}{c} x - x^3 + y = 0 \\ -y = 0 \end{array} \Leftrightarrow \begin{array}{c} x - x^3 = 0 \\ y = 0 \end{array} \Leftrightarrow \begin{array}{c} x(1-x)(1+x) = 0 \\ y = 0 \end{array}$$

$$\boxed{(-1,0),\ (0,0),\ \text{and}\ (1,0)}$$

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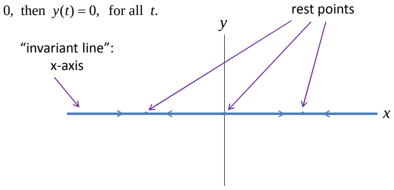
Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

If $y(0) = 0$, then $y(t) = 0$, for all t.



How do we analyze the full system?

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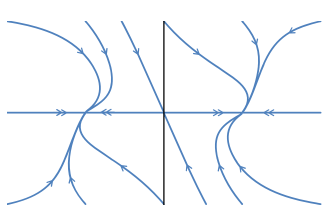
Dynamical Systems

Nonlinear Systems

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

Preview



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Dynamical Systems

Nonlinear Systems

Jacobian Matrix

$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2)$$

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^2, \quad f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\boxed{Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1, x_2) & \frac{\partial f_1}{\partial x_2}(x_1, x_2) \\ \frac{\partial f_2}{\partial x_1}(x_1, x_2) & \frac{\partial f_2}{\partial x_2}(x_1, x_2) \end{bmatrix}}$$

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Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

$$Df(x,y) = \begin{bmatrix} 1-3x^2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$Df(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

Dynamical Systems

Nonlinear Systems

$\frac{dx}{dt} = f(x)$ Rest point p: $f(p) = 0$

Linear approximation:

$$f(x) \approx f(p) + Df(p)(x-p) = Df(p)(x-p)$$

Introduce $\xi = x - p$.

Then $f(x) = f(p+\xi) \approx Df(p)\xi$

$$\frac{d\xi}{dt} = \frac{dx}{dt} = f(x) = f(p+\xi) \approx Df(p)\xi$$

Basic Idea

If ξ is small, i.e., if x is close to p,

then solutions of $\frac{d\xi}{dt} = f(\xi + p)$

are close to solutions of $\frac{d\xi}{dt} = Df(p)\xi$.

In particular, the rest point p is asymptotically stable for $\frac{dx}{dt} = f(x)$

if the origin is asymptotically stable for $\frac{d\xi}{dt} = Df(p)\xi$.

Dynamical Systems

Nonlinear Systems

$\frac{dx}{dt} = f(x)$ Rest point p: $f(p) = 0$

Criteria

The rest point p is saddle for $\frac{dx}{dt} = f(x)$ if $\det(Df(p)) < 0$.

It is a source if $\det(Df(p)) > 0$ and $\operatorname{trace}(Df(p)) > 0$.

It is a source if $\det(Df(p)) > 0$ and $\operatorname{trace}(Df(p)) < 0$.

Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

Rest points: $(-1,0)$, $(0,0)$, and $(1,0)$

$$Df(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad Df(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

$\det(Df(-1,0)) = 2 > 0$

$\operatorname{trace}(Df(-1,0)) = -3 < 0$

sink

$\operatorname{discriminant}(Df(-1,0)) = (-3)^2 - 4 \cdot 2 = 1 > 0$

stable node

$\det(Df(0,0)) = -1 < 0$

saddle

Dynamical Systems

Nonlinear Systems

Example

$$\frac{dx}{dt} = x - x^3 + y$$
$$\frac{dy}{dt} = -y$$

If $y(0) = 0$, then $y(t) = 0$, for all t.