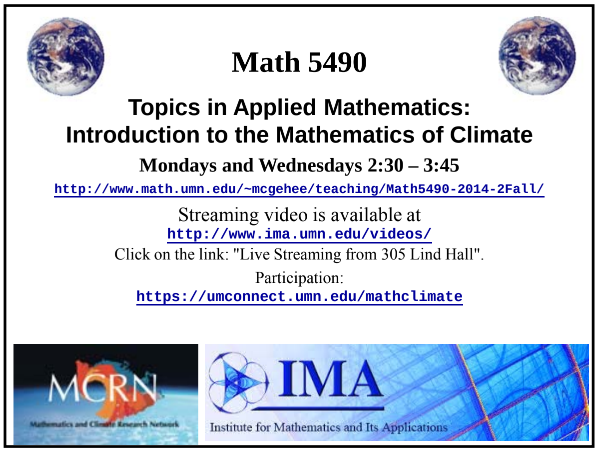


Math 5490

Topics in Applied Mathematics: Introduction to the Mathematics of Climate

Mondays and Wednesdays 2:30 – 3:45

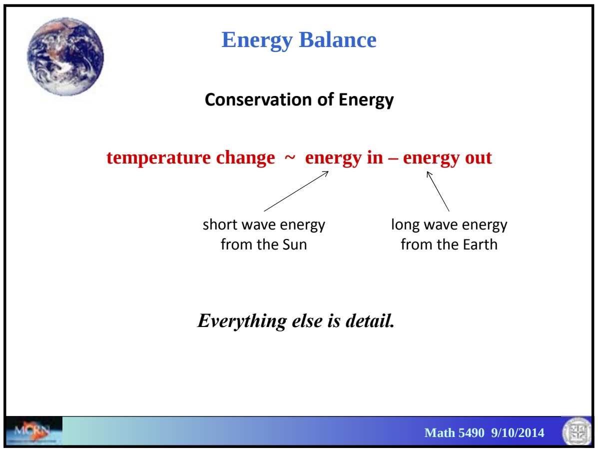
http://www.math.umn.edu/~mcgehee/teaching/Math5490-2014-2Fall/

Streaming video is available at
http://www.ima.umn.edu/videos/
Click on the link: "Live Streaming from 305 Lind Hall".

Participation:
https://umconnect.umn.edu/mathclimate

MCRN Mathematics and Climate Research Network

IMA Institute for Mathematics and Its Applications



Energy Balance

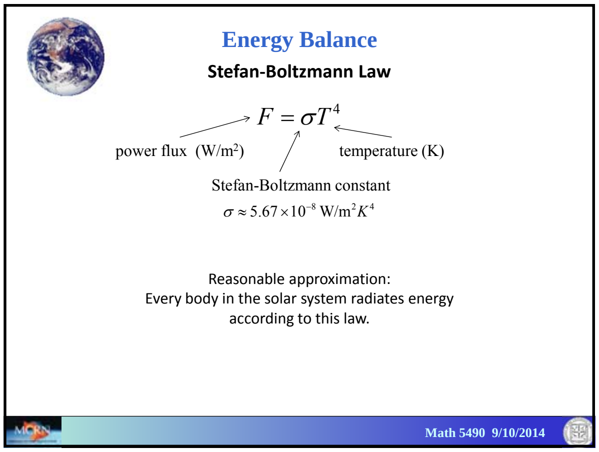
Conservation of Energy

temperature change ~ energy in – energy out

short wave energy from the Sun

long wave energy from the Earth

Everything else is detail.



Energy Balance

Stefan-Boltzmann Law

$$F = \sigma T^4$$

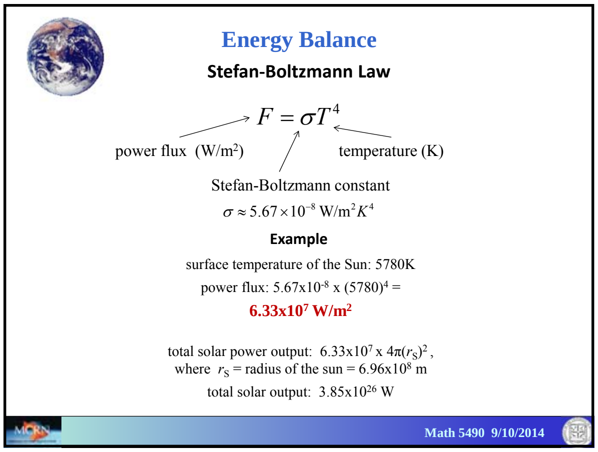
power flux (W/m^2)

temperature (K)

Stefan-Boltzmann constant

$$\sigma \approx 5.67 \times 10^{-8} \text{ W/m}^2 K^4$$

Reasonable approximation:
Every body in the solar system radiates energy according to this law.



Energy Balance

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$$F = \sigma T^4$$

power flux (W/m^2)

temperature (K)

Stefan-Boltzmann constant

$$\sigma \approx 5.67 \times 10^{-8} \text{ W/m}^2 K^4$$

Example

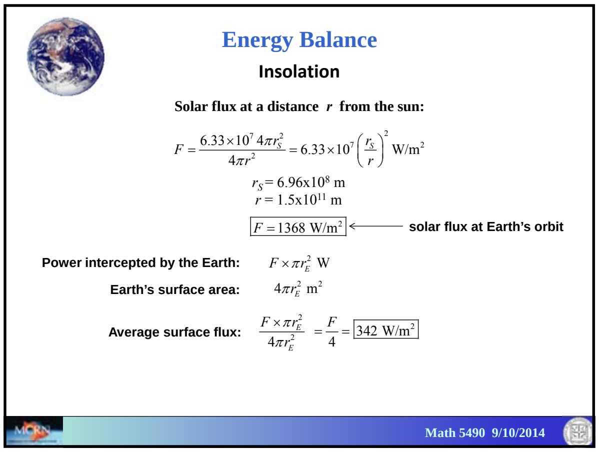
surface temperature of the Sun: 5780K

power flux: 5.67x10^{-8} x (5780)^4 =

6.33x10^7 W/m^2

total solar power output: 6.33x10^7 x 4π(r_S)^2 ,
where r_S = radius of the sun = 6.96x10^8 m

total solar output: 3.85x10^{26} W



Energy Balance

Insolation

Solar flux at a distance *r* from the sun:

$$F = \frac{6.33\times10^7\, 4\pi r_S^2}{4\pi r^2} = 6.33\times10^7 \left(\frac{r_S}{r}\right)^2 \text{ W/m}^2$$

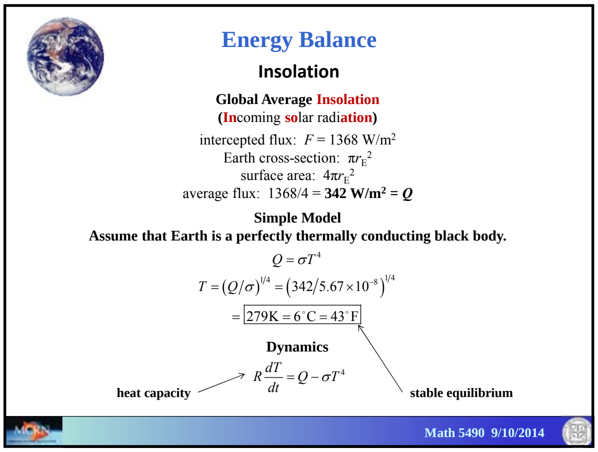
r_S = 6.96x10^8 m
r = 1.5x10^{11} m

$$\boxed{F = 1368 \text{ W/m}^2} \longleftarrow \text{solar flux at Earth's orbit}$$

Power intercepted by the Earth: $F \times \pi r_E^2$ W

Earth's surface area: $4\pi r_E^2$ m^2

Average surface flux: $\dfrac{F \times \pi r_E^2}{4\pi r_E^2} = \dfrac{F}{4} = \boxed{342 \text{ W/m}^2}$



Energy Balance

Insolation

Global Average Insolation
(Incoming solar radiation)

intercepted flux: F = 1368 W/m^2
Earth cross-section: πr_E^2
surface area: 4πr_E^2
average flux: 1368/4 = **342 W/m^2 = *Q***

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

$$Q = \sigma T^4$$

$$T = (Q/\sigma)^{1/4} = \left(342/5.67\times10^{-8}\right)^{1/4}$$

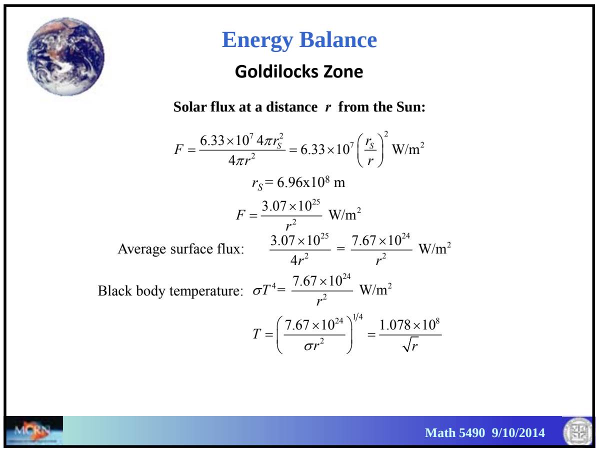
$$= \boxed{279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q - \sigma T^4$$

heat capacity

stable equilibrium



Energy Balance

Goldilocks Zone

Solar flux at a distance *r* from the Sun:

$$F = \frac{6.33\times10^7\, 4\pi r_S^2}{4\pi r^2} = 6.33\times10^7\left(\frac{r_S}{r}\right)^2 \text{ W/m}^2$$

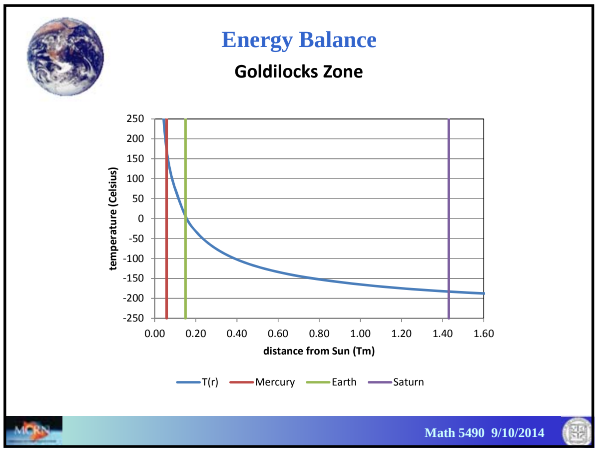
$$r_S = 6.96\text{x}10^8 \text{ m}$$

$$F = \frac{3.07\times10^{25}}{r^2} \text{ W/m}^2$$

Average surface flux: $\frac{3.07\times10^{25}}{4r^2} = \frac{7.67\times10^{24}}{r^2}$ W/m^2

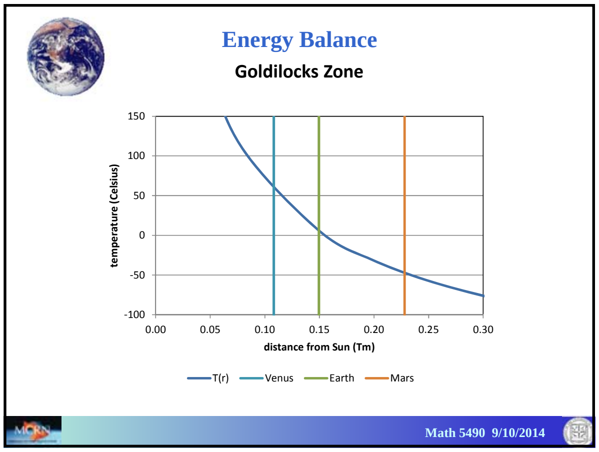
Black body temperature: $\sigma T^4 = \frac{7.67\times10^{24}}{r^2}$ W/m^2

$$T = \left(\frac{7.67\times10^{24}}{\sigma r^2}\right)^{1/4} = \frac{1.078\times10^8}{\sqrt{r}}$$



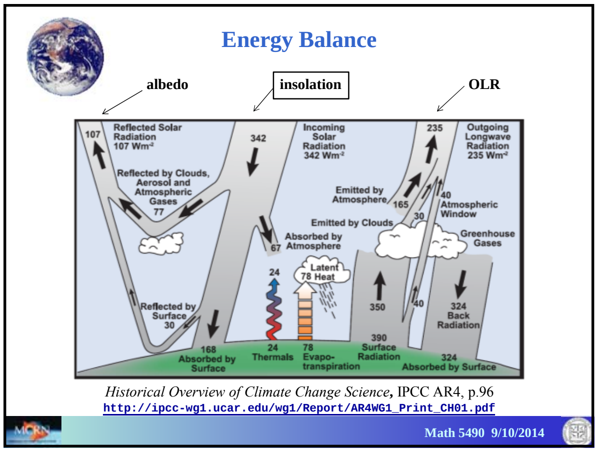
Energy Balance

Goldilocks Zone



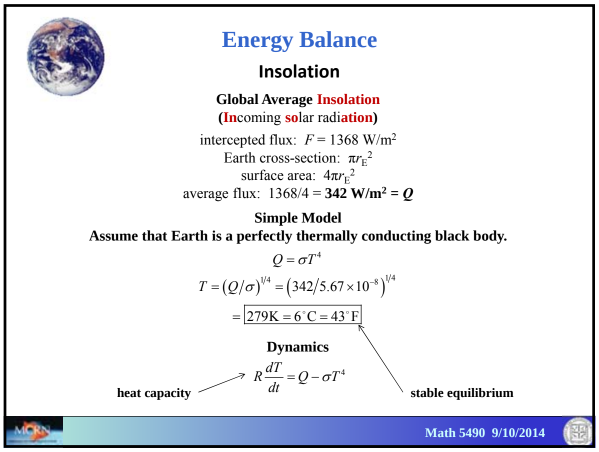
Energy Balance

Goldilocks Zone



Energy Balance

Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf



Energy Balance

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(Incoming solar radiation)

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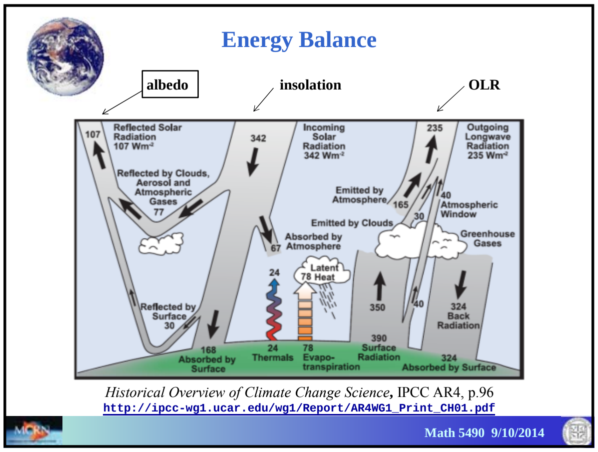
Dynamics

$$R\frac{dT}{dt} = Q - \sigma T^4$$

heat capacity

stable equilibrium



Energy Balance

Historical Overview of Climate Change Science, IPCC AR4, p.96
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Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.

The proportion reflected is called the albedo, denoted α .

For Earth, $\alpha \approx 0.3$.

Simple Model

Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = \left(0.7 \cdot F/\sigma\right)^{1/4} = \left(0.7 \cdot 342 / 5.67 \times 10^{-8}\right)^{1/4}$$

$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

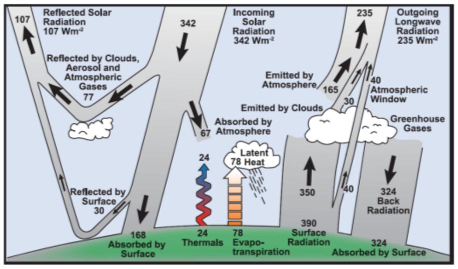
Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

Energy Balance

albedo **insolation** **OLR**



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Energy Balance

OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

$$\text{OLR} = A + BT$$

A and B are determined from satellite observations.

T is surface temperature (*in Celsius*).

$A = 202\ \text{W/m}^2$

$B = 1.90\ \text{W/m}^2\text{K}$

Dynamics

Kelvin

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

becomes

Celsius

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature

Energy Balance

OLR as a Function of Surface Temperature

$$\text{OLR} = A + BT$$

Important:

A+BT is **not** a linear approximation to the Stefan-Boltzmann equation.

different

Dynamics

Kelvin

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

becomes

Celsius

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature

Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

Equilibrium Temperature: $Q\left(1-\alpha\right) - A - BT_{eq} = 0$

$$T_{eq} = \frac{Q\left(1-\alpha\right) - A}{B}$$

Stable, since $B > 0$.

Ice-free planet: $\alpha = 0.32$, $T_{\text{eq}} = 16$ °C

Snowball planet: $\alpha = 0.62$, $T_{\text{eq}} = -38$ °C

No glacier would form on an ice-free Earth.

No glacier would melt on a snowball Earth.

Easy question:

Why do we have ice caps?

Hard question:

If Earth was ever a snowball, how did we get out?

Energy Balance

Latitude Dependence

Make T depend on $y = \sin(\text{latitude})$

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation distribution

$Q = \text{global annual average insolation} = 342\text{W/m}^2$

$s(y) = \text{distribution across latitudes} \quad \left(\int_0^1 s(y)dy = 1\right)$

One can show that

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\,d\theta$$

β = obliquity = 23.5°

Chylek and Coakley's quadratic approximation:

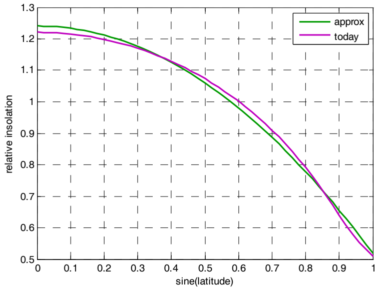
$$s\left(y\right) \approx 1 - 0.241\left(3y^2 - 1\right)$$

Energy Balance

Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.5°



Energy Balance

Latitude Dependence

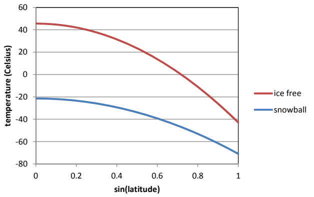
$$R\frac{\partial T(y,t)}{\partial t}=Qs(y)(1-\alpha)-\left(A+BT(y,t)\right)$$

Note that y is just a parameter.

Equilibrium Temperature Profile

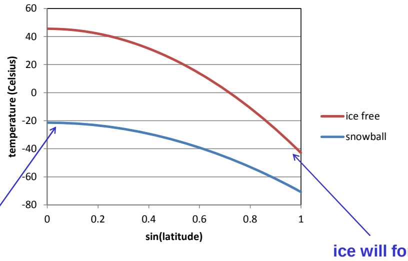
$$T_{eq}(y)=\frac{Qs(y)(1-\alpha)-A}{B}$$

$\alpha = 0.32$: ice free
$\alpha = 0.62$: snowball



Energy Balance

Latitude Dependence



ice won't melt
(no exit from snowball)

ice will form
(icecap)

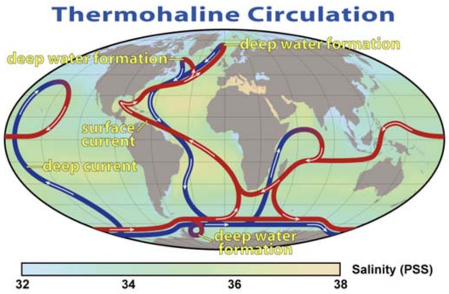
Energy Balance

What's Missing?

$$R\frac{\partial T(y,t)}{\partial t}=Qs(y)(1-\alpha)-\left(A+BT(y,t)\right)$$

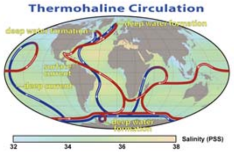
Energy Balance

What's Missing?



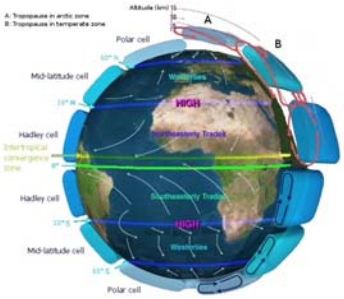
Energy Balance

What's Missing?



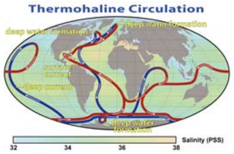
Energy Balance

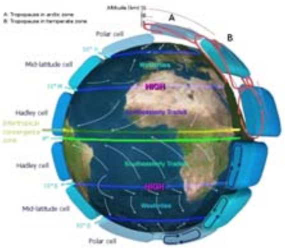
What's Missing?



Energy Balance

What's Missing?





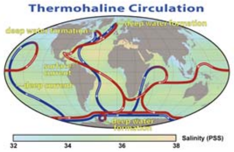
Energy Balance

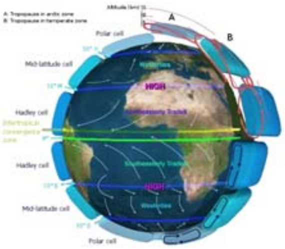
What's Missing?

Energy Balance

What's Missing?

Weather!





Energy Balance

Budyko's Model

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + \boxed{C(\overline{T} - T)}$$

Weather

global mean temperature $\overline{T}(t) = \int_0^1 T(y,t)\,dt$

Second Law of Thermodynamics:
Energy travels from hot places to cold places.

Budyko's equation as a dynamical system:
T lives in a function space (temperature as a function of latitude).

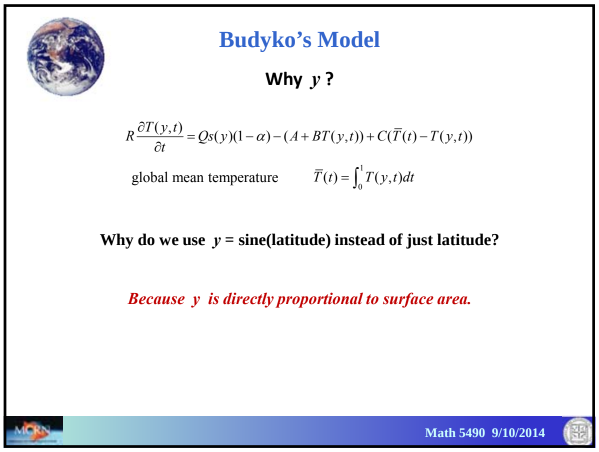
Budyko's Model

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A+BT(y,t)) + C(\overline{T}(t) - T(y,t))$$

global mean temperature: $\overline{T}(t) = \int_0^1 T(y,t)\,dt$

Why do we use y = sine(latitude) instead of just latitude?



Budyko's Model

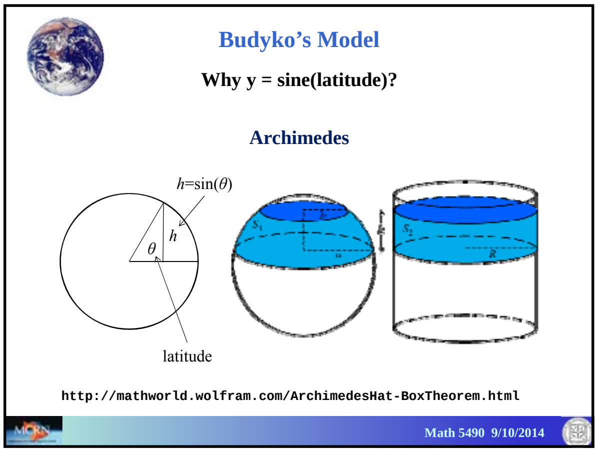
Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t)) + C(\overline{T}(t) - T(y,t))$$

global mean temperature $\overline{T}(t) = \int_0^1 T(y,t)dt$

Why do we use y = sine(latitude) instead of just latitude?

Because y is directly proportional to surface area.

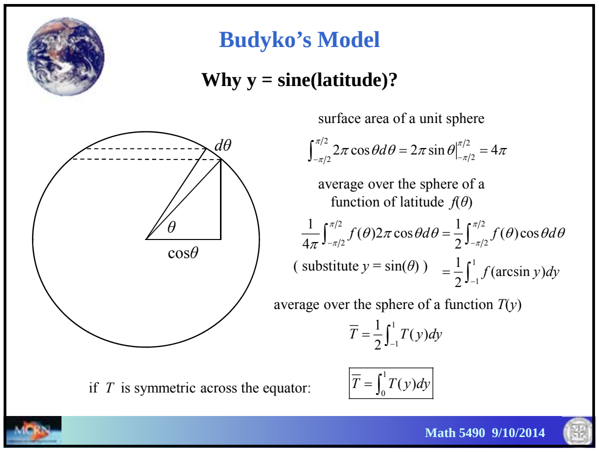


Budyko's Model

Why y = sine(latitude)?

Archimedes

http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html



Budyko's Model

Why y = sine(latitude)?

surface area of a unit sphere

$$\int_{-\pi/2}^{\pi/2} 2\pi\cos\theta d\theta = 2\pi\sin\theta\Big|_{-\pi/2}^{\pi/2} = 4\pi$$

average over the sphere of a function of latitude $f(\theta)$

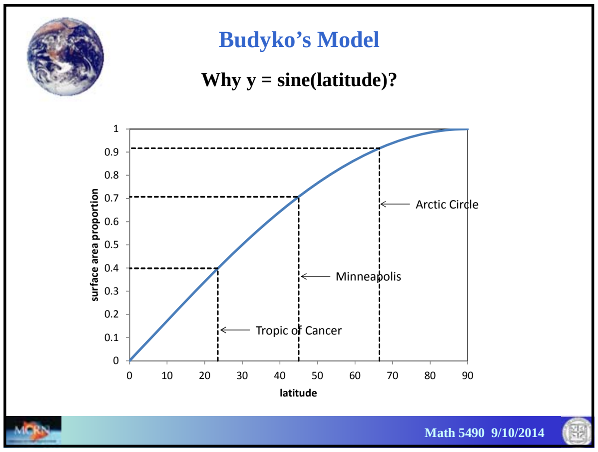
$$\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} f(\theta)2\pi\cos\theta d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} f(\theta)\cos\theta d\theta$$

(substitute $y = \sin(\theta)$) $= \frac{1}{2}\int_{-1}^{1} f(\arcsin y)dy$

average over the sphere of a function $T(y)$

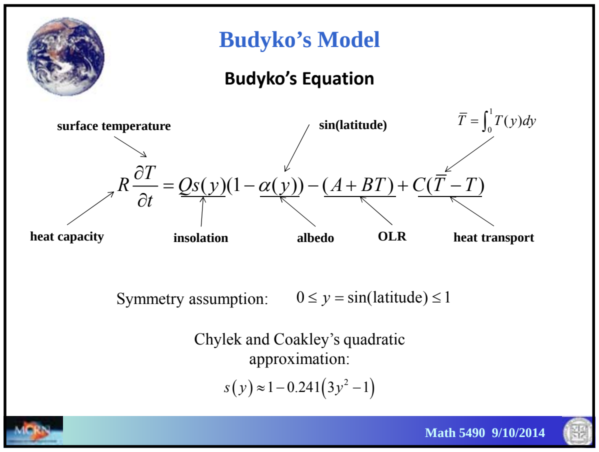
$$\overline{T} = \frac{1}{2}\int_{-1}^{1} T(y)dy$$

if T is symmetric across the equator: $\boxed{\overline{T} = \int_0^1 T(y)dy}$



Budyko's Model

Why y = sine(latitude)?



Budyko's Model

Budyko's Equation

surface temperature

sin(latitude)

$\overline{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = \underline{Qs(y)}(1-\underline{\alpha(y)}) - \underline{(A+BT)} + \underline{C(\overline{T}-T)}$$

heat capacity **insolation** **albedo** **OLR** **heat transport**

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241(3y^2 - 1)$$